FINAL REPORT

submitted to

CR-176563

NASA Goddard Space Flight Center
Greenbelt, MD

"Error Control Techniques for Satellite and Space Communications"

Grant NAG5-234
July 1, 1984 - December 31, 1985

Principal Investigator:

Daniel J. Costello, Jr.
Dept. of Electr. & Comp. Engr.
Univ. of Notre Dame*
Notre Dame, IN 46556

March, 1986

ERROR CONTROL TECHNIQUES FOR SATELLITE AND SPACE COMMUNICATIONS Final Report, 1 Jul. 1984 - 31 Dec. 1985 (Notre Dame Univ.) 103 p HC A06/MF A01

N86-20652

CSCL 17B

G3/32

Unclas
05653

*Formerly with Illinois Institute of Technology
Chicago, IL 60616

SUMMARY

During the grant period the following research results were obtained:

1) Performance Analysis of NASA Telecommand System.

Details of our work on this problem have been previously submitted to NASA. Three papers have been published in the open literature [1, 2, 3]. Copies of these papers have been sent to NASA in earlier reports. A complete summary of this work is included as Appendix A of this report. This constitutes Chapter 3 of the Ph.D. thesis of Dr. Robert H. Deng, who was supported as a research assistant on this grant and received his Ph.D. from the Illinois Institute of Technology in December, 1985.

2) Optimum Code Rate Selection in FEC Systems.

A random coding approach was taken to determine the optimum code rate to use in a forward-error-correcting (FEC) system with a fixed signal-energy-to-noise-power-density ratio, E_b/N_0, but no bandwidth constraint. By optimal code rate we mean the code rate which gives the smallest decoding error probability, or equivalently, the largest coding gain, for a given E_b/N_0. A paper is being prepared on this subject for submission to the IEEE Transactions on Information Theory [4]. A complete summary of this work is included as Appendix B of this report. This constitutes Chapter 4 of the Ph.D. thesis of Dr. Robert H. Deng.

Our results indicate, as expected, that when maximum likelihood decoding is used, the optimal code rate approaches zero (infinite bandwidth expansion). However, for more practical bounded distance decoders, an optimal code rate does exist between about a rate of 0.2 and a rate of 0.5 over a broad range for values for E_b/N_0. Calculations of code performance for several classes of specific codes tend to support these conclusions.

3) Capacity and Cutoff Rates of Concatenated Coding Systems.

Details of our work on this problem have been previously submitted to NASA. Four papers have been published in the open literature [5, 6, 7, 8]. Copies of these papers have been sent to NASA in earlier reports.

4) Distance Growth Rates in Convolutional Codes.

The rate of growth of the minimum distance between unmerged codewords in a convolutional code is an important parameter in determining the bit error probability of the code when used with a finite memory decoder. Also, if the code is terminated to form a block code, the performance of the block code depends on the distance between unmerged codewords in the convolutional code. We have obtained a lower bound on the minimum distance growth rate between unmerged codewords for time-invariant convolutional codes. This complements a similar result previously obtained for time-varying codes. A paper summarizing this result was previously submitted to NASA [9].

5) Bandwidth Efficient Coded Modulation.

Details of our work on bandwidth efficient coded modulation have been previously submitted to NASA. A copy of a paper on this subject was sent to NASA in an earlier report [10]. We are continuing our research in this area under our current NASA grant NAG5-557.

Papers Published Under NASA Grant NAG5-234

1. "Undetected Error Probability and Throughput Analysis of a Concatenated Coding Scheme," Proc. Allerton Conf. on Commun., Cont., and Comput., pp. 535-544, Monticello, IL, Oct., 1984 (with R. H. Deng).

2. "Performance Analysis of a Concatenated Coding Scheme for Error Control," Proc. IEEE Glob. Telecomm. Conf., pp. 769-773, Atlanta, GA, Nov. 1984 (with T. Kasami and S. Lin).

3. "Reliability and Throughput Analysis of a Concatenated Coding Scheme," submitted to <u>IEEE Trans. on Commun.</u>, Dec. 1985 (with R. H. Deng).

4. "On the Optimal Code Rate in an FEC System," in preparation (with R. H. Deng).

5. "Cutoff Rate Calculations for the Outer Channel in a Concatenated Coding System," Proc. Conf. on Inform. Sciences and Systems, pp. 238-243, Princeton, NJ, March 1984 (with M. A. Herro and L. Hu).

6. "Capacity Calculations for the Outer Channel in a Concatenated Coding System," Proc. Conf. on Inform. Sciences and Systems, p. 254, Baltimore, MD, March 1985 (with M. A. Herro and L. Hu).

7. "Random Coding Bounds and Error Performance for Concatenated Coding Systems," IEEE Int. Symp. on Inform. Th., Brighton, UK, June 1985 (with M. A. Herro and L. Hu).

8. "Capacity and Cutoff Rate Calculations for a Concatenated Coding System," submitted to <u>IEEE Trans. on Inform. Th.</u>, July 1985 (with M. A. Herro and L. Hu).

9. "A Lower Bound on the Minimum Distance Growth Rate of Fixed Convolutional Codes," IEEE Int. Symp. on Inform. Th., Brighton, UK, June 1985 (with J. L. Massey).

10. "Multidimensional Coded PSK Systems Using Unit-Memory Trellis Codes," Proc. Allerton Conf. on Commun., Cont., and Comput., pp. 428-429, Monticello, IL, Oct. 1985 (with A. LaFanechere).

Appendix A

CHAPTER III

UNDETECTED ERROR PROBABILITY AND THROUGHPUT ANALYSIS OF A CONCATENATED CODING SCHEME

3.1. Description of the Scheme

Consider a concatenated coding scheme for error control on a binary symmetric channel (BSC). Two linear block codes, denoted C_f and C_b, are used. The inner code C_f, called the frame code, is an (n,k) systematic binary block code with minimum distance d_f. The frame code is designed to correct t or fewer errors and simultaneously detect λ ($\lambda > t$) or fewer errors, where $t + \lambda + 1 \leq d_f$ [2]. The outer code is an (n_b, k_b) binary block code with

$$n_b = mk, \tag{3.1}$$

where m, a positive integer, is the number of frames. The outer code is designed for error detection only.

The encoding of the concatenated code is achieved in two stages (see Figure 3.1). A message of k_b bits is first encoded into a codeword of n_b bits in the outer code C_b. Then this codeword is interleaved to depth m. After interleaving, the n_b-bit block is divided into m k-bit segments. Each k-bit segment is encoded into an n-bit word in the frame code C_f. This n-bit word is called a frame. The two dimensional block format is depicted in Figure 3.2.

Decoding consists of error correction and error detection on each frame and error detection on the m decoded k-bit segments. When a frame in a block is received, it is first decoded based on the frame code C_f. The n-k parity bits are then removed from the decoded frame. If there

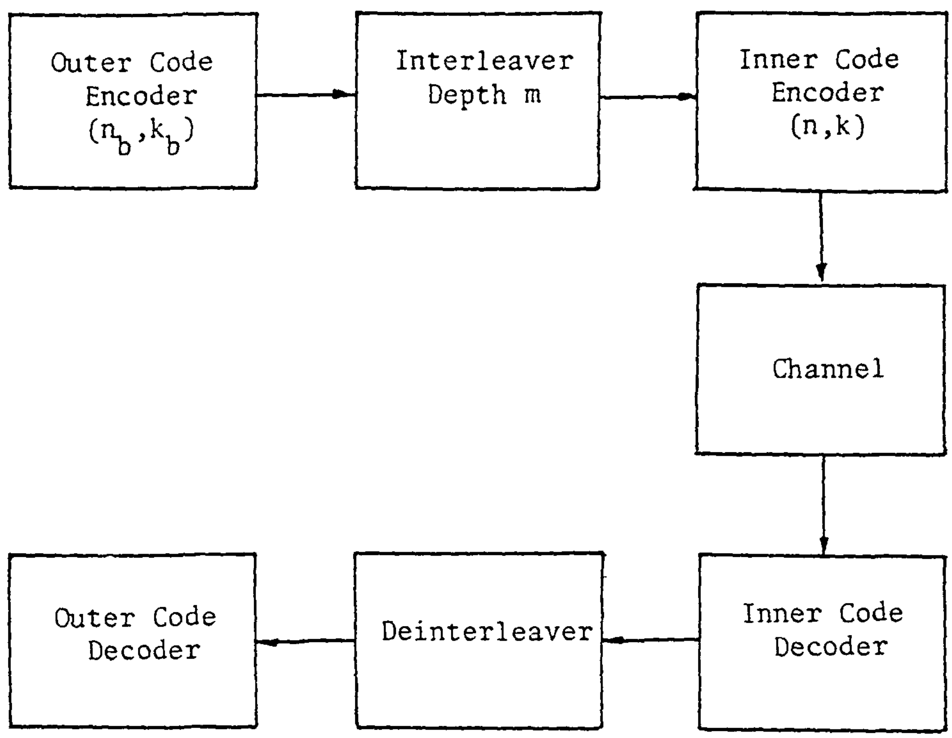

Figure 3.1. A Concatenated Coding System

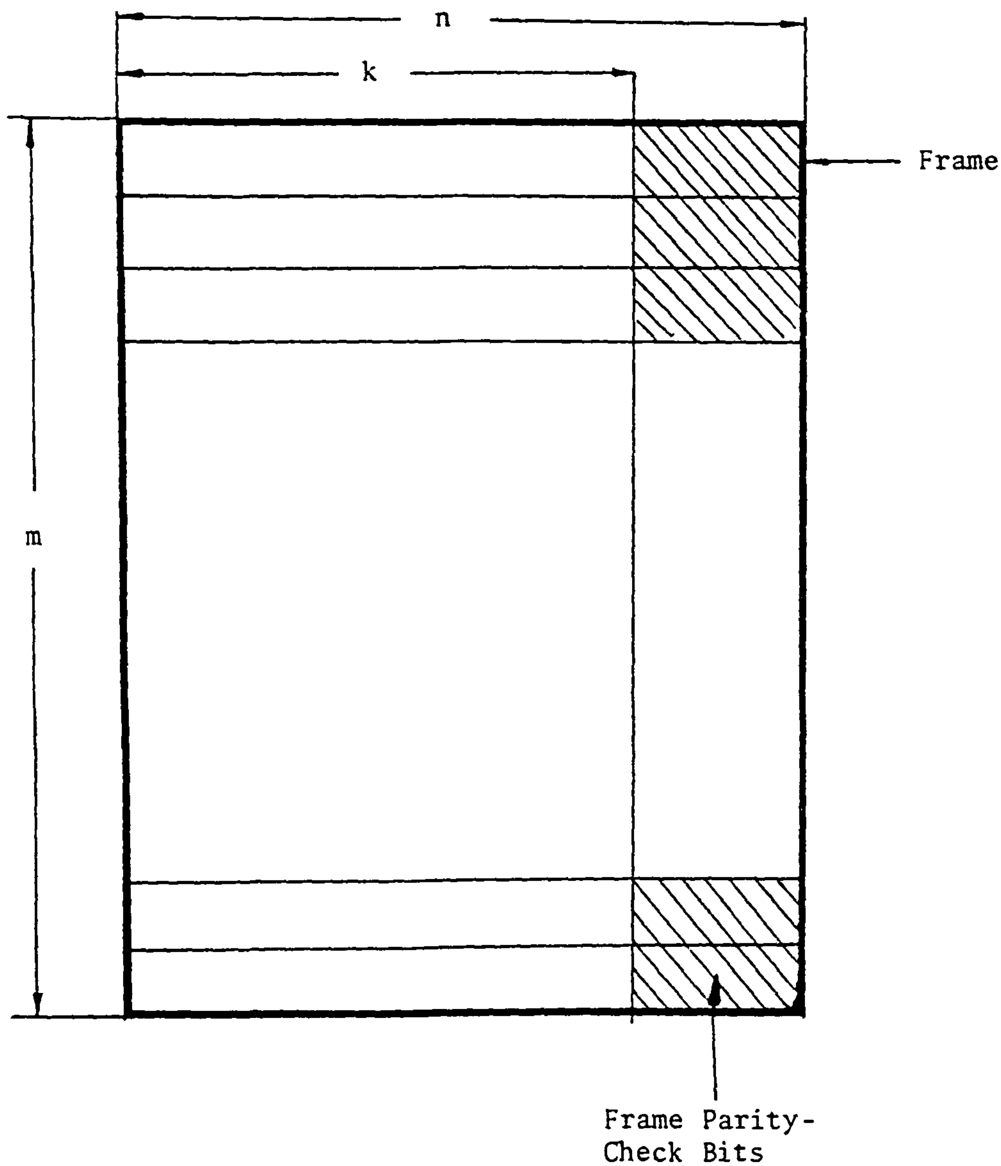

Figure 3.2. Block Format of a Concatenated Code Word

are t or fewer transmission errors in a received frame, the errors will be corrected, and the decoded segment is error free. If there are more than t errors in the received frame, the errors will be either detected or undetected. If the errors are detected, the decoder stops decoding immediately and requests a retransmission of the entire block. On the other hand, if the errors in a frame are undetected, the decoded segment will be stored in a buffer and the decoder continues to decode the next frame. After m frames of a block have been decoded, the m k-bit decoded segments are then deinterleaved. Error detection is performed on these deinterleaved segments based on the outer code C_b. If no errors are detected, the m decoded segments are assumed to be error free, and are accepted by the receiver. If the presence of errors is detected, the m decoded segments are discarded and the receiver requests a retransmission of the entire block.

The error control scheme described above is actually a combination of forward-error-correction (FEC) and automatic-repeat-request (ARQ). In this chapter, we analyze the performance of the proposed error control scheme. Specifically, the system reliability and the system throughput are considered. The system reliability is measured in terms of the probability of undetected error after decoding. First, by assuming the inner channel to be a memoryless binary symmetric channel (MBSC) with a bit error rate (BER) ε_i, we look at the outer channel created by the combination of the interleaver, the frame code, and the inner channel. Then we develop precise expressions for both the probability of undetected error and the system throughput. Following that, we investigate the system reliability attainable by using random coding arguments, which in turn are used as theoretical guidance in the selection of inner

and outer codes. Finally, the system performance on burst-noise-channels is considered.

3.2. The Outer Channel Model

Let $P_c^{(f)}(\varepsilon_i)$ denote the probability of correct decoding for the frame code. Suppose that a bounded-distance decoding algorithm is employed. Bounded-distance decoding corrects all received n-bit sequences with t or fewer errors. When an n-bit sequence with more than t errors is detected, no attempt is made to correct the errors. Since there are $\binom{n}{i}$ distinct ways in which i errors may occur among n bits,

$$P_c^{(f)}(\varepsilon_i) = \sum_{i=0}^{t} \binom{n}{i} \varepsilon_i^{\,i} (1-\varepsilon_i)^{n-i} \tag{3.2}$$

for bounded-distance decoding.

For a code word $\underline{v}$ in the frame code C_f, let $w(\underline{v})$ denote the Hamming weight of $\underline{v}$. If a decoded frame contains an undetectable error pattern, this error pattern must be a nonzero codeword in C_f. Let $\underline{e}_0$ be an undetectable error pattern after decoding. The probability $P_f(w,\varepsilon_i)$ that a decoded frame contains a nonzero error pattern $\underline{e}_0$ after decoding is given by [14,33,34]

$$P_f(w,\varepsilon_i) = \sum_{i=0}^{t} \sum_{j=0}^{\min(t-i,n-w)} \binom{w}{i}\binom{n-w}{j} \varepsilon_i^{\,w-i+j} (1-\varepsilon_i)^{n-w+i-j}, \tag{3.3}$$

where $w = w(\underline{e}_0)$, and ε_i is the BER of the inner channel. If $\varepsilon_i \ll \frac{1}{n}$, then

$$P_f(w,\varepsilon_i) \simeq \binom{w}{t} \varepsilon_i^{\,w-t} (1-\varepsilon_i)^{n-w+t}. \tag{3.4}$$

Let $P_{ud}^{(f)}(\varepsilon_i)$ denote the probability of undetected error for the frame

code. Let $\{A_w^{(f)}, d_f \leq w \leq n\}$ be the weight distribution of C_f. It follows from (3.3) and (3.4) that

$$P_{ud}^{(f)}(\varepsilon_i) = \sum_{w=d_f}^{n} A_w^{(f)} P_f(w,\varepsilon_i), \tag{3.5}$$

$$\begin{aligned} P_{ud}^{(f)}(\varepsilon_i) &\simeq A_{d_f}^{(f)} P_f(d_f,\varepsilon_i) \\ &\simeq A_{d_f}^{(f)} \binom{d_f}{t} \varepsilon_i^{d_f - t} (1-\varepsilon_i)^{n-d_f+t}, \quad \text{for } \varepsilon_i \ll \frac{1}{n}. \end{aligned} \tag{3.6}$$

Now consider any one of the m frames, say the j-th frame. If the decoded frame contains undetected errors, the BER ε_a after decoding is given by

$$\varepsilon_a = \frac{1}{n} \sum_{w=d_f}^{n} A_w^{(f)} w\ P_f(w,\varepsilon_i). \tag{3.7}$$

For $\varepsilon_i \ll \frac{1}{n}$, then

$$\begin{aligned} \varepsilon_a &\simeq \frac{1}{n} d_f\ A_{d_f}^{(f)} P_f(d_f,\varepsilon_i) \\ &\simeq \frac{d_f}{n} A_{d_f}^{(f)} \binom{d_f}{t} \varepsilon_i^{d_f - t} (1-\varepsilon_i)^{n-d_f+t}, \end{aligned} \tag{3.8}$$

is a good approximation to ε_a. Let E be defined as the event that a frame contains undetected errors. Now let $\varepsilon_{a/E}$ denote the BER embedded in a decoded frame conditioned on the occurrence of event E. It follows from (3.7) that

$$\varepsilon_{a/E} = \varepsilon_a / P_r\{E\} = \varepsilon_a / P_{ud}^{(f)}(\varepsilon_i). \tag{3.9}$$

For $\varepsilon_i \ll \frac{1}{n}$, substituting (3.6) and (3.8) into (3.9) yields

$$\varepsilon_{a/E} \simeq \frac{\frac{1}{n} d_f A^{(f)}_{d_f} P_f(d_f, \varepsilon_i)}{A^{(f)}_{d_f} P_f(d_f, \varepsilon_i)} = \frac{d_f}{n} . \tag{3.10}$$

Now define S to be a random variable such that when h of the m frames contain undetected errors, and the remaining m-h frames are decoded correctly, S = h, h = 0, 1, 2, $\cdots$, m. It follows from (3.2) and (3.5) that

$$P_r\{S=h\} = \binom{m}{h} \cdot [P^{(f)}_{ud}(\varepsilon_i)]^h \cdot [P^{(f)}_c(\varepsilon_i)]^{m-h}. \tag{3.11}$$

Note that (3.11) is not a binomial distribution because $P^{(f)}_{ud}(\varepsilon_i) + P^{(f)}_c(\varepsilon_i) < 1$, i.e., some received sequences with more than t errors are detected by the frame code.

After deinterleaving of the m decoded segments (with the n-k parity bits removed from each frame), the BER embedded in the n_b-bit block, conditioned on S = h, is given by

$$\varepsilon_0(h) = \varepsilon_{a/E} \cdot \frac{h}{m} \quad , \quad k = 0, 1, 2, \cdots, m. \tag{3.12}$$

We call the channel specified by (3.11) and (3.12) the outer channel, and it is depicted in Figure 3.3. Note that $\varepsilon_0(0) = 0$. This channel can be viewed as a <u>block interference</u> (BI) channel, as described in [35]. Δ_h, h = 0, 1, 2, $\cdots$, m, is called the h-th component channel of the BI channel. Each block of n_b bits (n_b is the length of the outer code) is transmitted over one of the m+1 component channels. The random variable S determines which component channel is used to transmit a given n_b-bit block.

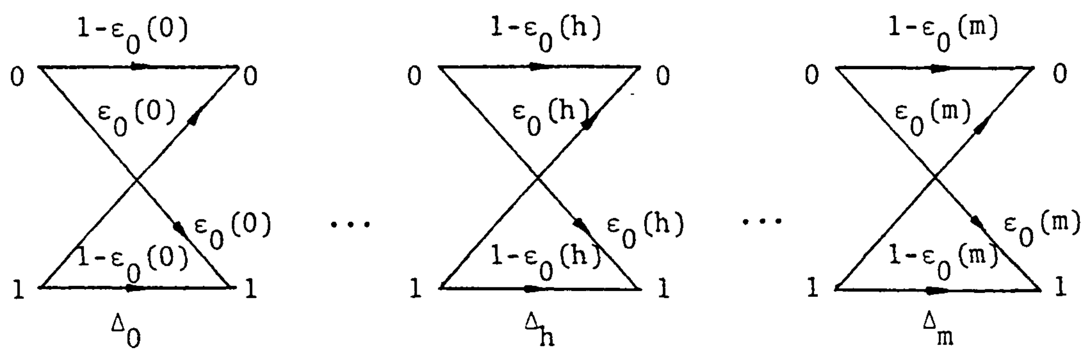

Figure 3.3. Outer Channel Resulting from Decoding Inner Code on an MBSC

3.3. The Probability of Undetected Error and the System Throughput of the Concatenated Code

Let $\{A_i^{(b)}, d_b \leq i \leq n_b\}$ be the weight distribution of the outer code, where d_b is the minimum distance of C_b. Let $P_{ud}^{(b)}(\varepsilon)$ be the probability of undetected error for the outer code C_b. If the n_b-bit block is transmitted over the h-th component channel Δ_h of the outer channel, it follows from (3.12) that

$$P_{ud}^{(b)}(\varepsilon_0(h)) = \sum_{i=d_b}^{n_b} A_i^{(b)}(\varepsilon_0(h))^i(1-\varepsilon_0(h))^{n_b-i} \tag{3.13}$$

Let $P_{ud}(\varepsilon_i)$ be the average probability of undetected error of the concatenated code. From (3.11) and (3.13) we obtain

$$\begin{aligned} P_{ud}(\varepsilon_i) &= \sum_{h=0}^{m} P_r\{S=h\}P_{ud}^{(b)}(\varepsilon_0(h)) \\ &= \sum_{h=1}^{m} \{\binom{m}{h}[P_{ud}^{(f)}(\varepsilon_i)]^h[P_c^{(f)}(\varepsilon_i)]^{m-h} \\ &\quad \cdot \sum_{i=d_b}^{n_b} A_i^{(b)}(\varepsilon_0(h))^i(1-\varepsilon_0(h))^{n_b-i}\}, \end{aligned} \tag{3.14}$$

where $P_c^{(f)}(\varepsilon_i)$ and $P_{ud}^{(f)}(\varepsilon_i)$ are given by (3.2) and (3.5), respectively.

The system throughput is defined as the ratio of the average number of information digits successfully accepted by the receiver per unit of time to the total number of digits that could be transmitted per unit of time [8,36]. It is determined by the retransmission strategy, which may be one of the three basic types: stop-and-wait, go-back-N, or selective-repeat. All three basic ARQ schemes achieve the same reliability; however, they have different throughputs. Suppose that the

selective-repeat ARQ scheme is used as the retransmission strategy. The specific manner in which the receiver signals to the transmitter for a retransmission will not be considered. It will be assumed, however, that this backward signal is error-free, and that repeated retransmissions of a block are possible. For an analysis of various ARQ schemes with a noisy feedback channel, the reader is referred to Reference 8.

For the concatenated code, let $P_{ud}(\varepsilon_i)$, $P_r(\varepsilon_i)$, and $P_c(\varepsilon_i)$ denote the probabilities of an undetected error, of a block retransmission, and of correct decoding, respectively. Obviously,

$$P_{ud}(\varepsilon_i) + P_r(\varepsilon_i) + P_c(\varepsilon_i) = 1. \tag{3.15}$$

In the selective-repeat ARQ scheme, the transmitter sends code blocks to the receiver continuously and resends only those code blocks that are detected in error at the receiver. The probability that a block will be accepted by the receiver is

$$P(\varepsilon_i) = P_{ud}(\varepsilon_i) + P_c(\varepsilon_i). \tag{3.16}$$

For a code block to be successfully accepted by the receiver, the average number of retransmissions (including the original transmission) required is

$$\begin{aligned} N &= 1\cdot P(\varepsilon_i) + 2\cdot P(\varepsilon_i)(1-P(\varepsilon_i)) + 3\cdot P(\varepsilon_i)(1-P(\varepsilon_i))^2 + \cdots + \\ &\quad + \ell\cdot P(\varepsilon_i)(1-P(\varepsilon_i))^{\ell-1} + \cdots \\ &= \frac{1}{P(\varepsilon_i)} = \frac{1}{P_{ud}(\varepsilon_i) + P_c(\varepsilon_i)}. \end{aligned} \tag{3.17}$$

Then the throughput of the system is [2]

$$\eta = R \cdot \frac{1}{N} = \frac{k}{n} \cdot \frac{k_b}{n_b} \cdot (P_{ud}(\varepsilon_i) + P_c(\varepsilon_i)), \tag{3.18}$$

where $R = \frac{k}{n} \cdot \frac{k_b}{n_b}$ is the over-all code rate of the concatenated code.

Note that a transmitted block will be received correctly if and only if all m frames are decoded correctly. Therefore, the probability of accepting a correct block is given by

$$P_c(\varepsilon_i) = [P_c^{(f)}(\varepsilon_i)]^m = [\sum_{i=0}^{t} \binom{n}{i}\varepsilon_i^{\,i}(1-\varepsilon_i)^{n-i}]^m. \quad (3.19)$$

For the usual situation where $P_{ud}(\varepsilon_i) << P_c(\varepsilon_i)$, it follows from (3.18) and (3.19) that

$$\eta \approx \frac{k}{n} \cdot \frac{k_b}{n_b} \cdot [\sum_{i=0}^{t} \binom{n}{i}\varepsilon_i^{\,i}(1-\varepsilon_i)^{n-i}]^m. \quad (3.20)$$

It can easily be seen that η increases monotonically as t increases, but that for small ε_i, η is only a weakly increasing function of t.

In order to see the relationship between t and $P_{ud}(\varepsilon_i)$, from (3.14) we have

$$P_{ud}(\varepsilon_i) \simeq m \cdot P_{ud}^{(f)}(\varepsilon_i) \cdot [P_c^{(f)}(\varepsilon_i)]^{m-1}$$

$$\cdot \{\sum_{i=d_b}^{n_b} A_i^{(b)}(\varepsilon_0(1))^i(1-\varepsilon_0(1))^{n_b-i}\}, \text{ for } \varepsilon_i << \frac{1}{n}. \quad (3.21)$$

Using (3.6), (3.10), and (3.12), $P_{ud}(\varepsilon_i)$ can be further approximated as

$$P_{ud}(\varepsilon_i) \simeq K \cdot \binom{d_f}{t}\varepsilon_i^{\,d_f-t}(1-\varepsilon_i)^{n-d_f+t} \cdot [P_c^{(f)}(\varepsilon_i)]^{m-1}, \quad (3.22)$$

where

$$K = m \cdot A_{d_f}^{(f)} \cdot \{\sum_{i=d_b}^{n_b} A_i^{(b)}(\frac{d_f}{m\cdot n})^i(1 - \frac{d_f}{m\cdot n})^{n_b-i}\}$$

is a constant which is independent of t. Let Q(t) denote the right hand side of (3.22). Then

$$\frac{Q(t+1)}{Q(t)} \simeq \frac{(d_f - t)}{(t+1)} \cdot \frac{1}{\varepsilon_i} >> n \quad , \quad \text{for } \varepsilon_i << \frac{1}{n} . \tag{3.23}$$

That is, for $\varepsilon_i << 1/n$, when t increases by 1, $P_{ud}(\varepsilon_i)$, the probability of undetected error, will increase by approximately ε_i^{-1}. Thus $P_{ud}(\varepsilon_i)$ is a strongly increasing function of t. For this reason, a large value of t is not desirable in such a system.

3.4. A Bound on the Reliability of the Concatenated Code by a Random Coding Argument

In this section we derive a lower bound on the reliability of the concatenated code by using a random coding approach. Although the bound may not be tight enough, especially when the inner code is used for error detection only (i.e., t=0), it does give some insight into the concatenated code.

Let $P_E^{(f)}(\varepsilon_i)$ be the probability of decoding error for the frame code when the frame code is used only for error correction. Again, let $P_{ud}^{(f)}(\varepsilon_i)$ and $P_c^{(f)}(\varepsilon_i)$ be the probabilities of undetected error after error correction and of correct decoding for the frame code, respectively. Obviously,

$$P_{ud}^{(f)}(\varepsilon_i) \leq P_E^{(f)}(\varepsilon_i). \tag{3.24}$$

Then $P_{ud}(\varepsilon_i)$, the probability of undetected error of the concatenated code, from (3.14) and (3.24), can be bounded by

$$P_{ud}(\varepsilon_i) = \sum_{h=1}^{m} \binom{m}{h} [P_{ud}^{(f)}(\varepsilon_i)]^h [P_c^{(f)}(\varepsilon_i)]^{m-h} P_{ud}^{(b)}(\varepsilon_0(h)) \; .$$

$$\leq \sum_{h=1}^{m} \binom{m}{h} [P_{ud}^{(f)}(\varepsilon_i)]^h P_{ud}^{(b)}(\varepsilon_0(h))$$

$$\leq \sum_{h=1}^{m} \binom{m}{h} [P_{E}^{(f)}(\varepsilon_i)]^h P_{ud}^{(b)}(\varepsilon_0(h)). \tag{3.25}$$

Let Γ_1 and Γ_2 denote the ensembles of inner systematic codes and of outer systematic codes, respectively. We assume that the two ensembles of codes are selected independently of each other. Then the average probability of undetected error over $\Gamma_1 \times \Gamma_2$ is, from (3.25)

$$\overline{P_{ud}(\varepsilon_i)} \leq \overline{\sum_{h=1}^{m} \binom{m}{h} [P_{E}^{(f)}(\varepsilon_i)]^h P_{ud}^{(b)}(\varepsilon_0(h))},$$

$$= \overline{\sum_{h=1}^{m} \binom{m}{h} [P_{E}^{(f)}(\varepsilon_i)]^h} \cdot \overline{P_{ud}^{(b)}(\varepsilon_0(h))}, \tag{3.26}$$

where the first average is over Γ_1 and the second over Γ_2. Equation (3.26) can be rewritten as

$$\overline{P_{ud}(\varepsilon_i)} \leq \sum_{h=1}^{m} \binom{m}{h} \overline{[P_{E}^{(f)}(\varepsilon_i)]^h} \; \overline{P_{ud}^{(b)}(\varepsilon_0(h))}, \tag{3.27}$$

because the average of the sum is equal to the sum of the averages. By the memoryless assumption of the inner channel, we also have, from (3.27),

$$\overline{P_{ud}(\varepsilon_i)} \leq \sum_{h=1}^{m} \binom{m}{h} \overline{[P_{E}^{(f)}(\varepsilon_i)]}^h \cdot \overline{P_{ud}^{(b)}(\varepsilon_0(h))}. \tag{3.28}$$

The last term in (3.28) is given by [2]

$$\overline{P_{ud}^{(b)}(\varepsilon_0(h))} \le 2^{-(n_b-k_b)} = 2^{-n_b(1-R_2)}, \tag{3.29}$$

which is independent of h, where $R_2 = k_b/n_b$ is the code rate of the outer code. Using this fact, equation (3.28) yields

$$\overline{P_{ud}(\varepsilon_i)} \le 2^{-n_b(1-R_2)} \sum_{h=1}^{m} \binom{m}{h} [\overline{P_E^{(f)}(\varepsilon_i)}]^h. \tag{3.30}$$

Now we define the <u>reliability</u> of an (n,k) code as

$$E \ge - \lim_{n\to\infty} \frac{1}{n} \log_2 \overline{P_{ud}}. \tag{3.31}$$

For the concatenated code, the reliability is

$$E \ge - \lim_{n_b\to\infty} \frac{1}{n_b} \log_2 \overline{P_{ud}(\varepsilon_i)}. \tag{3.32}$$

Note that $n_b = km = R_1 mn$. For fixed R_1 and m, n_b going to infinity is equivalent to n going to infinity. Hence (3.32) can be rewritten as

$$\begin{aligned} E \ge &- \lim_{n_b\to\infty} \frac{1}{n_b} \log_2 2^{-n_b(1-R_2)} \\ &- \lim_{n\to\infty} \frac{1}{mR_1 n} \log_2 \{\sum_{h=1}^{m} \binom{m}{h} [\overline{P_E^{(f)}(\varepsilon_i)}]^h\} \\ &= 1-R_2 + \frac{1}{mR_1} \{- \lim_{n\to\infty} \frac{1}{n} \log_2 \overline{P_E^{(f)}(\varepsilon_i)} \\ &- \lim_{n\to\infty} \frac{1}{n} \log_2 [m + \sum_{h=2}^{m} \binom{m}{h} [\overline{P_E^{(f)}(\varepsilon_i)}]^{h-1} . \end{aligned} \tag{3.33}$$

Because $\overline{P_E^{(f)}(\varepsilon_i)} \to 0$ when $n\to\infty$ if and only if $R_1 < C$, where $C = 1 - H(\varepsilon_i) = 1 + \varepsilon_i \log_2 \varepsilon_i + (1-\varepsilon_i)\log_2(1-\varepsilon_i)$ is the inner channel capacity [15], the last limit in the brackets $\{\cdots\}$ goes to zero. Therefore

$$E \geq 1-R_2 + \frac{1}{mR_1}\{ - \lim_{n\to\infty} \frac{1}{n} \log_2 \overline{P_E^{(f)}(\varepsilon_i)} \}$$

$$= 1-R_2 + \frac{1}{mR_1} E_f, \tag{3.34}$$

where $E_f \triangleq - \lim_{n\to\infty} \frac{1}{n} \log_2 \overline{P_E^{(f)}(\varepsilon_i)}$ is given by [15]

$$E_f = \begin{cases} \lambda \log_2 \dfrac{1}{\sqrt{4\varepsilon_i(1-\varepsilon_i)}} & , \quad 0 \leq R_1 \leq R_b, \\ R_0 - R_1 & , \quad R_b \leq R_1 \leq R_c, \\ E(\lambda, \varepsilon_i) & , \quad R_c \leq R_1 \leq C, \end{cases} \tag{3.35}$$

where

$$\lambda = H^{-1}(1-R_1), \tag{3.36}$$

$$E(\lambda,\varepsilon_i) = H(\varepsilon_i) + (\lambda-\varepsilon_i)H'(\varepsilon_i) - H(\lambda), \tag{3.37}$$

$$H'(\varepsilon_i) = \frac{d}{d\varepsilon_i} H(\varepsilon_i), \tag{3.38}$$

$$R_b = 1 - H\left(\frac{\sqrt{4(\varepsilon_i(1-\varepsilon_i)}}{1+\sqrt{4\varepsilon_i(1-\varepsilon_i)}} \right), \tag{3.39}$$

$$R_c = 1 - H\left(\frac{\sqrt{\varepsilon_i}}{\sqrt{\varepsilon_i} + \sqrt{1-\varepsilon_i}} \right), \tag{3.40}$$

and

$$R_0 = 1 - \log_2(1 + 2\sqrt{\varepsilon_i(1-\varepsilon_i)}). \tag{3.41}$$

The reliability E, bounded in (3.34) is plotted in Figures 3.4 and 3.5 for $\varepsilon_i = 10^{-2}$ and $\varepsilon_i = 10^{-4}$, respectively. These results show that low inner and outer code rates are needed in order to obtain high system reliability.

3.5. Examples

Having investigated the theory of the concatenated code in the previous sections, in this section we present some concatenated code examples whose purpose is to give a feeling for the actual system performance. Recall that the concatenated coding scheme described above is used in ARQ systems, and that the major advantage of ARQ is that it requires simple decoding equipment, while achieving high system reliability and throughput. Therefore, only those codes which require simple decoding are chosen as examples. We should point out that the results of sections 3.3-3.4 are a useful general guide to code selection.

Example 3.1

This concatenated code example has been proposed for a NASA telecommand system. The frame code C_f is a distance-4 Hamming code with generator polynomial

$$g(x) = (x+1)(x^6+x+1) = x^7 + x^6 + x^2 + 1, \tag{3.42}$$

where $x^6 + x + 1$ is a primitive polynomial of degree 6. The natural length of this code is 63. This code is used for single error correction (t=1), and is also used to detect all error patterns of double weight and some higher odd weight error patterns. The outer code is a

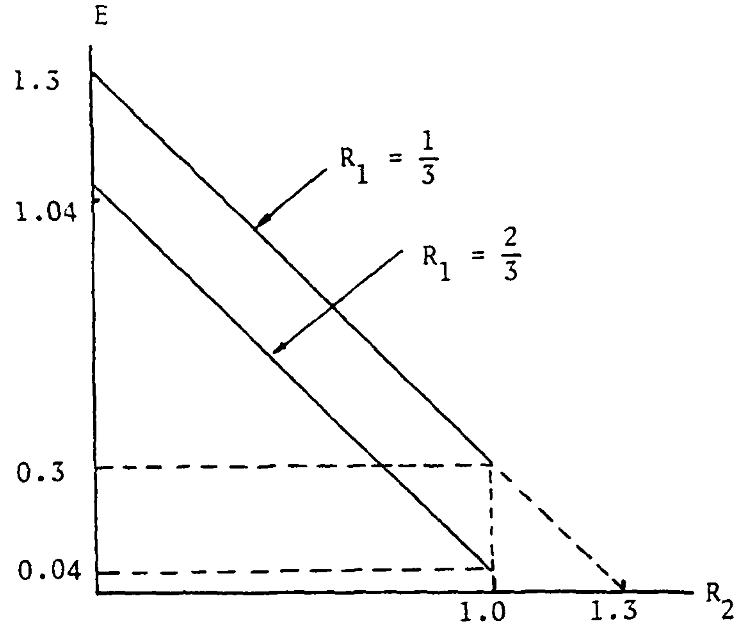

Figure 3.4. Reliability of the Concatenated Code with m=4, $\varepsilon_i = 10^{-2}$

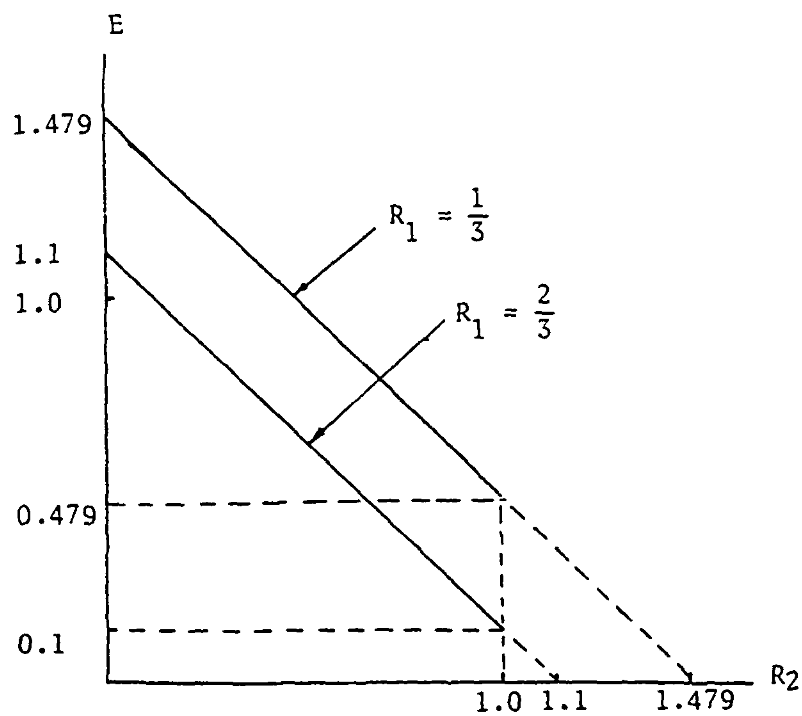

Figure 3.5. Reliability of the Concatenated Code with m=4, $\varepsilon_i = 10^{-4}$

distance-4 shortened Hamming code with generator polynomial

$$g(x) = (x+1)(x^{15} + x^{14} + x^{13} + x^{12} + x^4 + x^3 + x^2 + x + 1)$$
$$= x^{16} + x^{12} + x^5 + 1, \qquad (3.43)$$

where $x^{15} + x^{14} + x^{13} + x^{12} + x^4 + x^3 + x^2 + x + 1$ is a primitive polynomial of degree 15. This code is the x.25 standard for packet-switched data networks [37]. The natural length of this code is $2^{15}-1 = 32{,}767$. In this example, a shortened code of maximum length 3,584 bits is considered. This code is used for error detection only.

We assume that the number of information bytes (IB) in a frame is between 3 and 7, that is, the inner code can also be shortened. The number of frames in a block is between 4 and 64.

To obtain a precise result for $P_{ud}(\varepsilon_i)$, a computer program was written to help determine the reliability of the proposed concatenated coding scheme. We found that if only one frame contains a weight 4 undetected error pattern, then this error pattern can always be detected by the outer code. Thus (3.14) can be modified as follows:

$$\begin{aligned} P_{ud}(\varepsilon_i) = {} & \binom{m}{1}\overline{P}_{ud}^{(f)}(\varepsilon_i)P_c^{(f)}(\varepsilon_i)^{m-1} \\ & \cdot \sum_{i=d_b}^{n_b} A_i^{(b)}(\overline{\varepsilon}_0(1))^i(1-\overline{\varepsilon}_0(1))^{n_b-i} \\ & + \sum_{h=2}^{m}\{\binom{m}{h}[P_{ud}^{(f)}(\varepsilon_i)]^h[P_c^{(f)}(\varepsilon_i)]^{m-h} \\ & \cdot \sum_{i=d_b}^{n_b} A_i^{(b)}(\varepsilon_0(h))^i(1-\varepsilon_0(h))^{n_b-i}\}, \end{aligned} \qquad (3.44)$$

where

$$\overline{P}^{(f)}(\varepsilon_i) = \sum_{w=d_f+1}^{n} A_w^{(f)} P_f(w,\varepsilon_i), \tag{3.45.1}$$

and

$$\overline{\varepsilon}_0(1) = \frac{\frac{1}{n} \sum_{w=d_f+1}^{n} A_w^{(f)} w \; P_f(w,\varepsilon_i)}{\overline{P}_{ud}^{(f)}(\varepsilon_i)} \cdot \frac{1}{m}. \tag{3.45.2}$$

Results for the probability of undetected error $P_{ud}(\varepsilon_i)$, based on (3.44), and the system throughput η are plotted in Figure 3.6, where we have used the method in [6] to obtain

$$P_{ud}^{(b)}(\varepsilon_0(h)) = \sum_{i=d_b}^{n_b} A_i^{(b)} (\varepsilon_0(h))^i (1-\varepsilon_0(h))^{n_b-i}$$

The system described above can be altered by allowing the frame code to do error detection only (i.e., t=0). In this case, $P_{ud}(\varepsilon_i)$ and η are shown in Figure 3.7.

Example 3.2

The same frame code and outer code are employed as in Example 3.1. The inner channel is, however, assumed to be a AWGN channel with BPSK modulation and the frame code is decoded by using the Viterbi decoding algorithm with repeat request and infinite demodulator output quantization [38]. Let u, a positive real number, be the retransmission metric threshold of the algorithm [38]. Let $P_{ud}^{(f)}$, $P_d^{(f)}$, and ε_a denote the probability of undetected error, the probability of detected error, and the BER after decoding, respectively, for the frame code. Then [38]

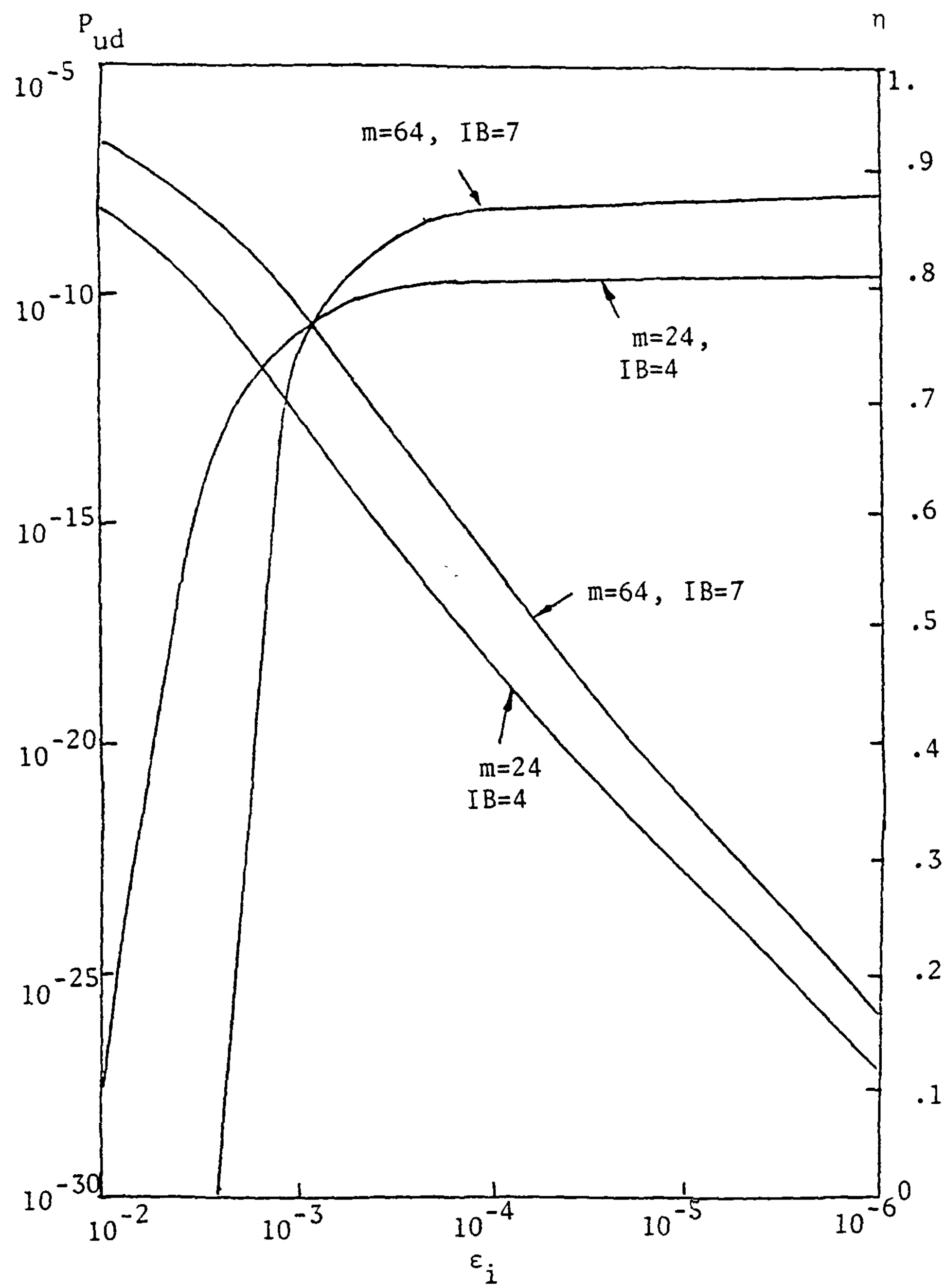

Figure 3.6. Performance of the Concatenated Code of Example 3.1 (t=1)

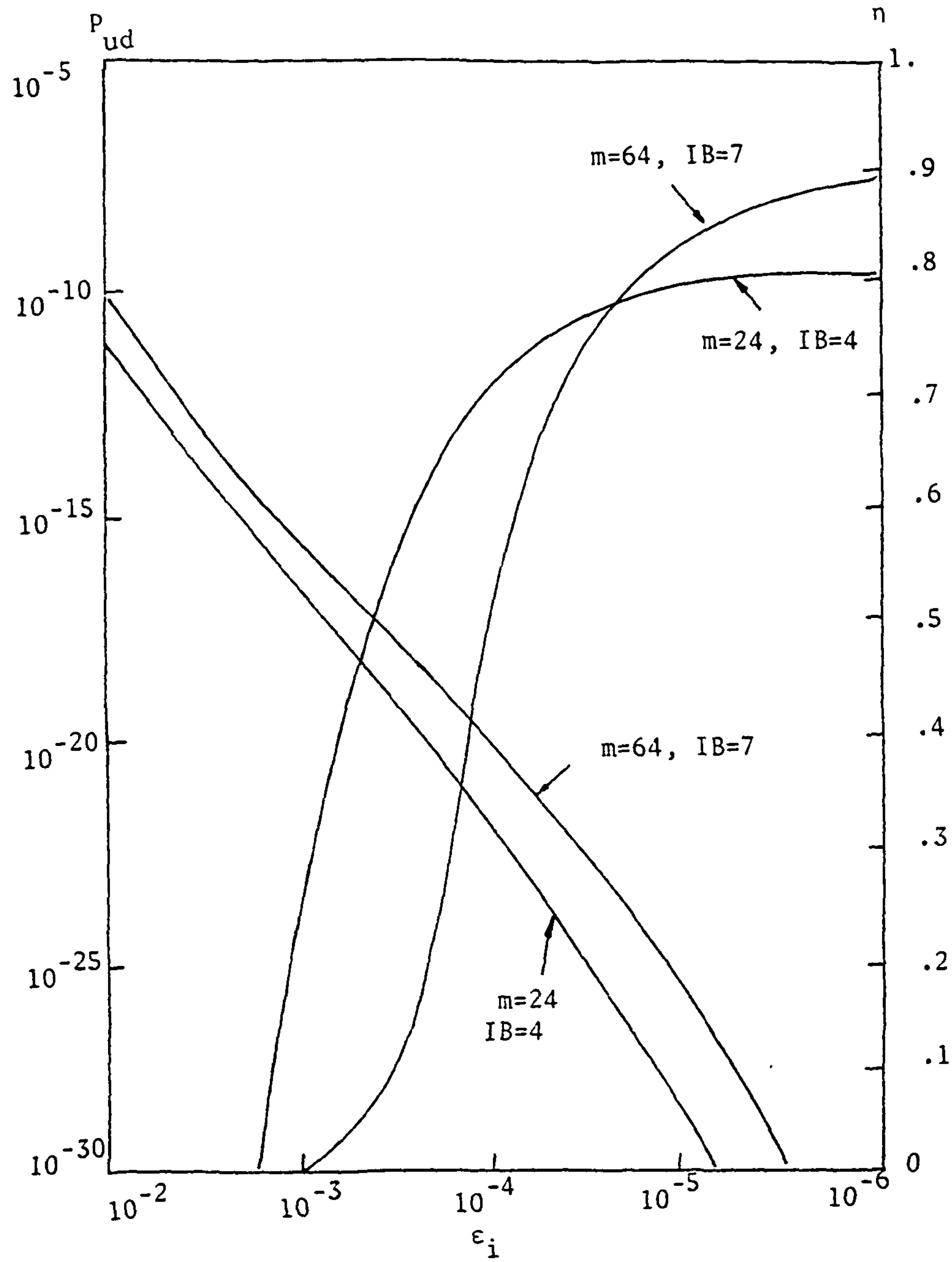

Figure 3.7. Performance of the Concatenated Code of Example 3.1 (t=0)

$$P_{ud}^{(f)} \leq kQ\left(\sqrt{\frac{2E_N}{N_0} d_f + u\sqrt{\frac{2E_N}{N_0}}}\right)$$

$$\cdot \exp\left(\frac{E_N}{N_0} d_f\right) T(X) \Big|_{X=\exp\left(-\frac{E_N}{N_0}\right)} , \tag{3.46}$$

$$P_{d}^{(f)} \leq kQ\left(\sqrt{\frac{2E_N}{N_0} d_f - u\sqrt{\frac{2E_N}{N_0}}}\right)$$

$$\cdot \exp\left(\frac{E_N}{N_0} d_f\right) T(X) \Big|_{X=\exp\left(-\frac{E_N}{N_0}\right)} , \tag{3.47}$$

$$\varepsilon_a \leq Q\left(\sqrt{\frac{2E_N}{N_0} d_f + u\sqrt{\frac{2E_N}{N_0}}}\right)$$

$$\exp\left(\frac{E_N}{N_0} d_f\right) \frac{\partial T(X,Y)}{\partial Y} \Bigg|_{\substack{Y=1 \\ X=\exp\left(-\frac{E_N}{N_0}\right)}} , \tag{3.48}$$

where

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-z^2/2} dz, \tag{3.49}$$

$$T(X) = \sum_{i=d_f}^{n} A_i^{(f)} X^i, \tag{3.50}$$

$$\frac{\partial T(X,Y)}{\partial Y} \Bigg|_{Y=1} \simeq \sum_{i=d_f}^{n} i\, A_i^{(f)} X^i, \tag{3.51}$$

and E_N/N_0 is the channel symbol signal energy-to-noise power density ratio. From (3.46) and (3.47) we see that the probability of correct

decoding for the frame code is

$$P_c^{(f)} = 1 - P_{ud}^{(f)} - P_d^{(f)} \tag{3.52.1}$$

$$\approx 1 - P_d^{(f)} \quad , \quad \text{for} \quad P_{ud}^{(f)} << P_d^{(f)}. \tag{3.52.2}$$

The probability of undetected error of the concatenated code, P_{ud}, and the system throughput, η, can be computed by using (3.46)-(3.52.2) in the formulas given in section 3.3. Both P_{ud} and η are shown in Figure 3.8 for u=4 with respect to ε_i, where E_N/N_0 and ε_i are related by the equation

$$\varepsilon_i = Q\left(\sqrt{\frac{2E_N}{N_0}} \right). \tag{3.53}$$

The influence of the value of u on the system performance is obvious. For larger values of u, from (3.46), (3.47), and (3.52.2), the probabilities $P_{ud}^{(f)}$ and $P_c^{(f)}$ become smaller, and consequently the probability of undetected error and the system throughput are lower.

Example 3.3

The outer code is again a shortened distance-4 Hamming code with generator polynomial given by (3.43). The frame code is an (n,n-1) single-parity-check code. The frame code has a minimum distance of 2, and is used for error detection only. The frame code can detect all odd weight error patterns. The weight distribution of the frame code can be calculated from

$$A_{2i} = \binom{n-1}{2i} + \binom{n-1}{2i-1} \quad , \quad i = 0, 1, 2, \cdots \lfloor \tfrac{n}{2} \rfloor , \tag{3.54.1}$$

$$A_j = 0, \quad \text{for all odd } j, \tag{3.54.2}$$

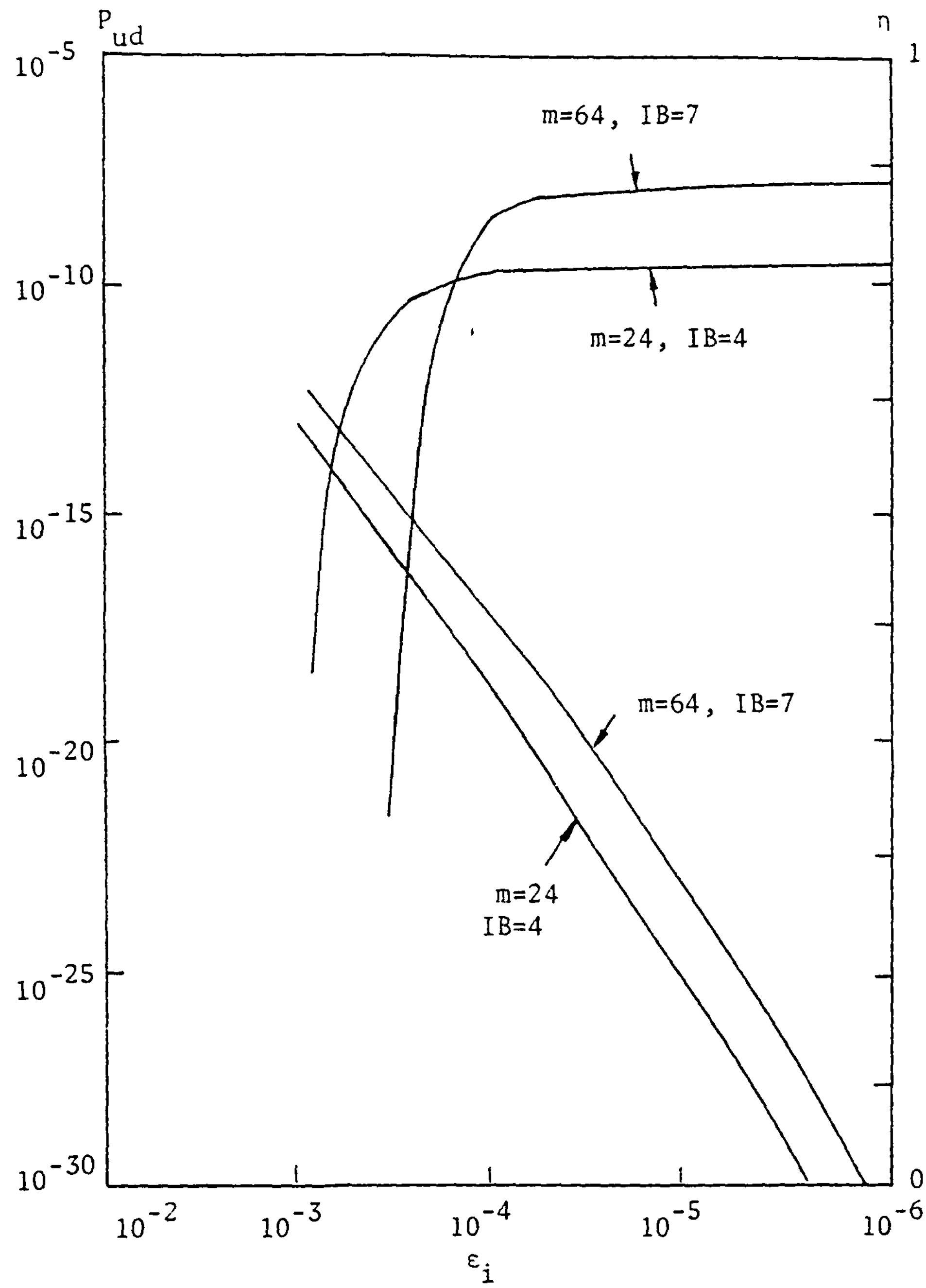

Figure 3.8. Performance of the Concatenated Code of Example 3.2 with u=4

where $\binom{n}{k} = 0$ for $k < 0$ and $k > n$, and $\lfloor x \rfloor$ denotes the integer part of x.

Because the outer code detect three or fewer errors, if only one frame contains a weight 3 or less undetected error pattern, then this error pattern can always be detected by the outer code. Hence, equation (3.44) is used to compute the probability of undetected error, where

$$\overline{P}_{ud}^{(f)}(\varepsilon_i) = \sum_{i=2}^{\lfloor \frac{n}{2} \rfloor} A_{2i}\varepsilon_i^{2i}(1-\varepsilon_i)^{n-2i} \tag{3.55.1}$$

is the probability of undetected error when the undetected error pattern has weight greater than 3, and

$$\overline{\varepsilon}_0(1) = \frac{\sum_{i=2}^{\lfloor \frac{n}{2} \rfloor} \frac{2i}{n} A_{2i}\varepsilon_i^{2i}(1-\varepsilon_i)^{n-2i}}{\overline{P}_{ud}^{(f)}(\varepsilon_i)} \cdot \frac{1}{m} . \tag{3.55.2}$$

Figure 3.9 shows the probability of undetected error $P_{ud}(\varepsilon_i)$ and the system throughput η for this example.

From Figures 3.6-3.9, we observe that the performance of a particular scheme depends strongly upon the channel noise conditions. Therefore, we cannot say that a particular one of the above schemes is "best". However, we can draw a number of conclusions which will be discussed below.

From Figures 3.6 and 3.7 we can see the tradeoffs between the probability of undetected error and the system throughput obtained by varying the number of correctable errors t in the frame code. Smaller value of t always result in a lower probability of undetected error,

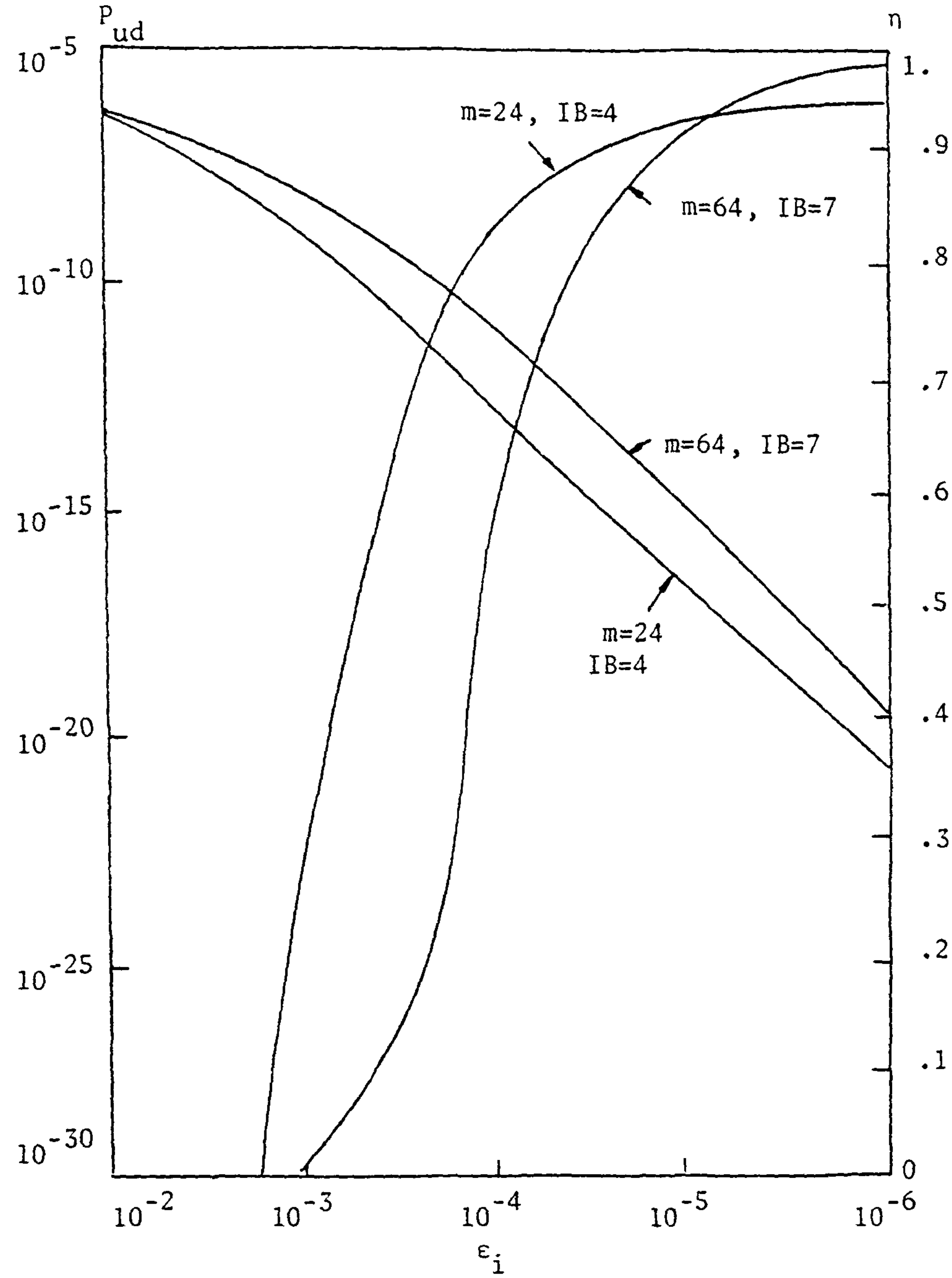

Figure 3.9. Performance of the Concatenated Code of Example 3.

and therefore a higher system reliability. But as the channel BER gets higher, the system throughput degrades rapidly for smaller t. The system throughput is less affected by t if the channel BER is small.

Figure 3.8 shows the advantages of a Viterbi decoded frame code with repeat request over an algebraically decoded frame code. The Viterbi decoding algorithm makes the system much more flexible in trading between system reliability and throughput by simply changing the value of u. Varying u can be viewed as a generalized method of "varying t" for algebric decoding of the frame code. From comparison of Figures 3.6-3.9 we see that lower inner code rates provide higher system reliabilities but lower system throughputs.

In Figure 3.10 we plot $P_{ud}(\varepsilon_i)$ vs. η for the above examples with $m = 64$ and $IB = 7$. The infinite slope of the curves is due to the fact that at low channel BER's the system throughput becomes saturated. We conclude that, at moderately low BER's, the concatenated coding scheme is capable of achieving high system throughputs and extremely low undetected error probabilities.

3.6. The Concatenated Code Performance on a Burst-Noise-Channel

Channels with memory often occur in practice. Errors on these channels tend to occur in bursts, and hence they are called burst-noise-channels. Examples of burst-noise-channels are radio channels, where the error bursts are caused by signal fading due to multipath transmission, wire and cable transmission, which is affected by impulsive switching noise and crosstalk, and magnetic recording, which is subject to tape dropouts due to surface defects and dust particles. In this

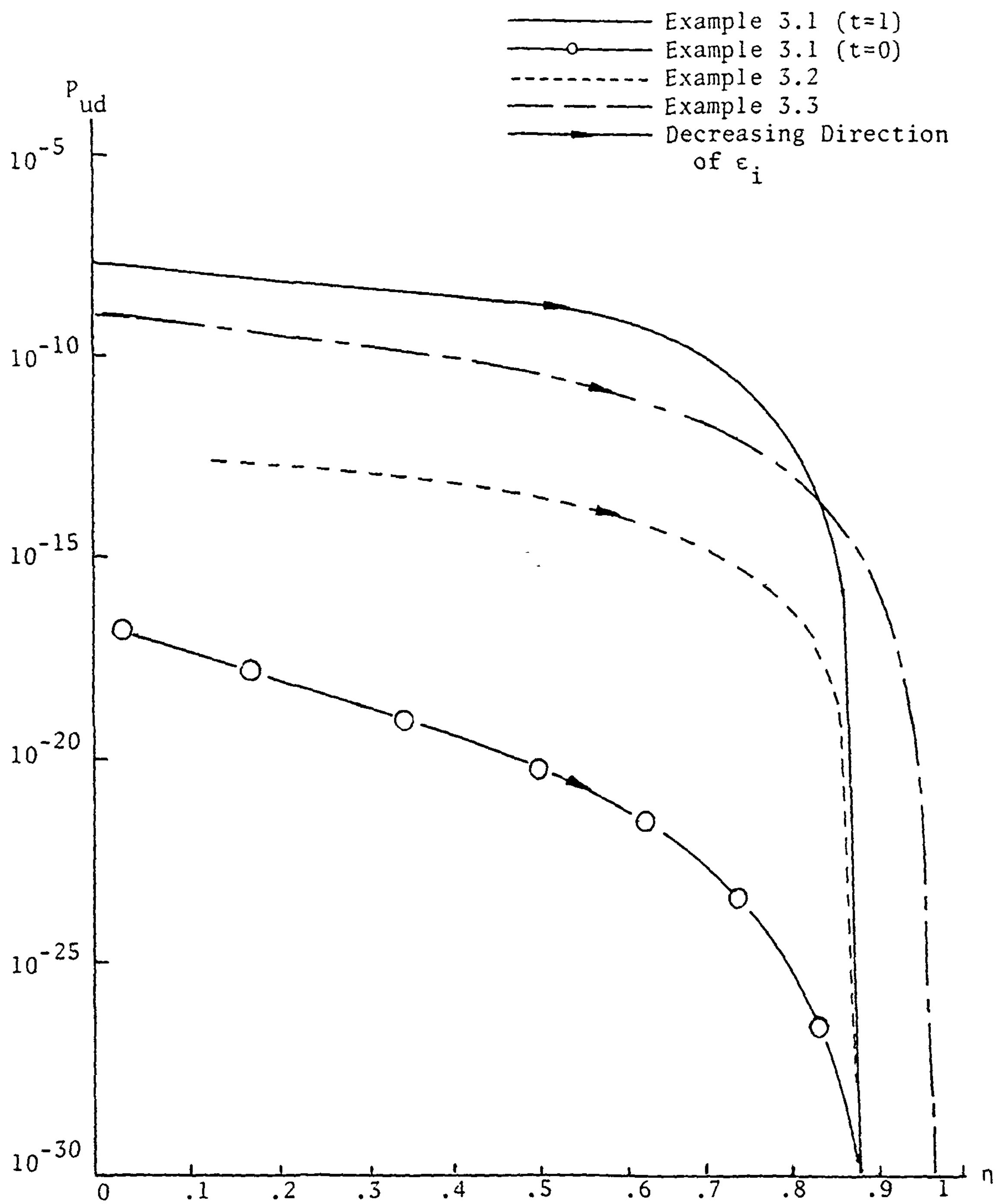

Figure 3.10. Probabilities of undetected Error vs. System Throughput of Examples 3.1-3.3

section we extend the performance analysis of the concatenated coding scheme to burst-noise-channels. The results here will be compared with those presented in the previous sections.

3.6.1. The Inner Channel Model

The generalized Gilbert type channel [11,39,40], as shown in Figure 3.11, is used as our inner channel model. There are two states in the model. Each state is a BSC. State 1 is the "quiet" state, where the BER is ε_1. State 2 is the "noisy" state, where the BER is ε_2, and $\varepsilon_2 >> \varepsilon_1$. The transition probabilities between states are $P = P_r\{1 \to 2\}$ and $p = P_r\{2 \to 1\}$ (see Figure 3.11). $Q = 1-P$ and $q = 1-p$ are the probabilities of remaining in states 1 and 2, respectively. To simplify the model's treatment, we assume that one transition time in the model corresponds to the transmission of one frame of length n bits, i.e., the noisy bursts last for a multiple of the transmission time of a frame. The average burst length is then [11]

$$\bar{L} = \frac{1}{p} \text{ frames,} \tag{3.56}$$

or

$$\bar{\ell} = \bar{L}n = \frac{1}{p} n \text{ bits.} \tag{3.57}$$

The average BER is

$$\bar{\varepsilon} = \frac{1}{P+p} (p\varepsilon_1 + P\varepsilon_2), \tag{3.58}$$

and the probability of being in the noisy state is

$$p_2 = \frac{\bar{\varepsilon} - \varepsilon_1}{\varepsilon_2 - \varepsilon_1}. \tag{3.59}$$

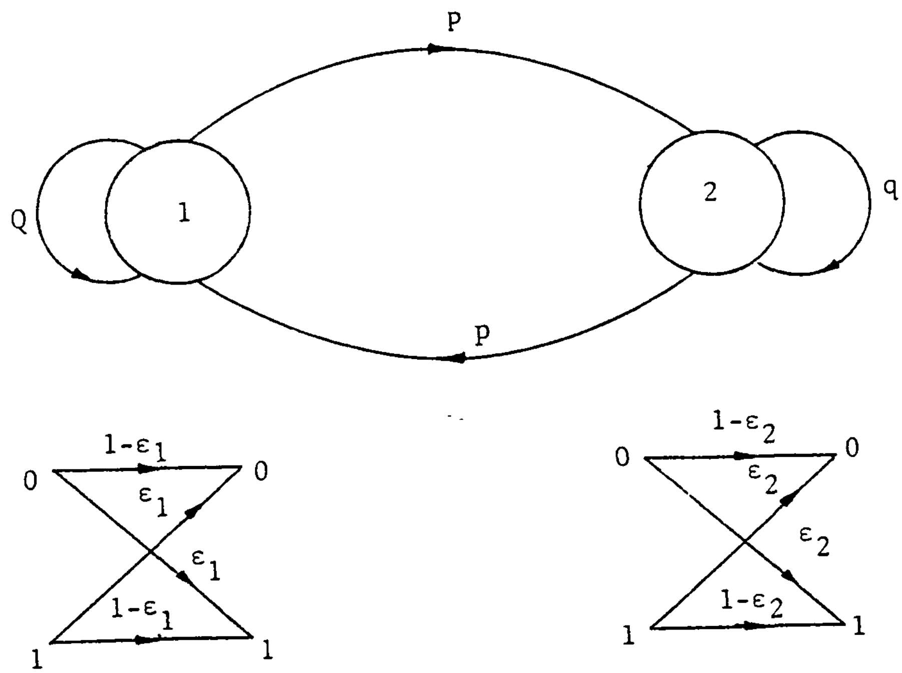

Figure 3.11. A Burst-Noise Inner Channel

Four parameters govern the model. They can be chosen to be $\bar{L}$, $\bar{\varepsilon}$, p_2, and the high-to-low BER ratio $\varepsilon_2/\varepsilon_1$.

3.6.2. The Outer Channel Model

Let $P_c^{(f)}(\varepsilon_j)$, $P_{ud}^{(f)}(\varepsilon_j)$, ε_{aj} and $\varepsilon_{aj/E}$, j = 1, 2, denote the probability of correct decoding for the frame code, the probability of undetected error for the frame code, the BER in a decoded frame, and the BER embedded in the decoded frame conditioned on the decoded frame containing undetected errors, respectively, when the frame is transmitted in state j. (In the following we will always use the subscript j, j = 1, 2, to denote that a frame is transmitted in state j.) Then $P_c^{(f)}(\varepsilon_j)$, $P_{ud}^{(f)}(\varepsilon_j)$, ε_{aj}, and $\varepsilon_{aj/E}$ are given by (3.2), (3.5), (3.7), and (3.9), respectively, with ε_i replaced by ε_j, j = 1, 2.

Now define $E_{\ell,h}$, $0 \leq \ell \leq h \leq m$, to be an event such that h of the m decoded frames contain undetected errors (the other m-h decoded frames are error free) and ℓ of the h containing undetected error frames are transmitted in state 2 of the inner channel. Let $P_r\{E_{\ell,h}\}$ be the probability that event $E_{\ell,h}$ occurs. Then after deinterleaving of the m segments (with the n-k parity bits removed from each decoded frame), the BER embedded in the n_b-bit block, conditioned on the occurrence of event $E_{\ell,h}$, is given by

$$\varepsilon_0(E_{\ell,h}) = [\ell \cdot \varepsilon_{a2/E} + (h-\ell)\varepsilon_{a1/E}]/m \quad , \quad 0 \leq \ell \leq h \leq m. \qquad (3.60)$$

We call the channel specified by (3.60) and the probability distribution $P_r\{E_{\ell,h}\}$ the outer channel (see Figure 3.12).

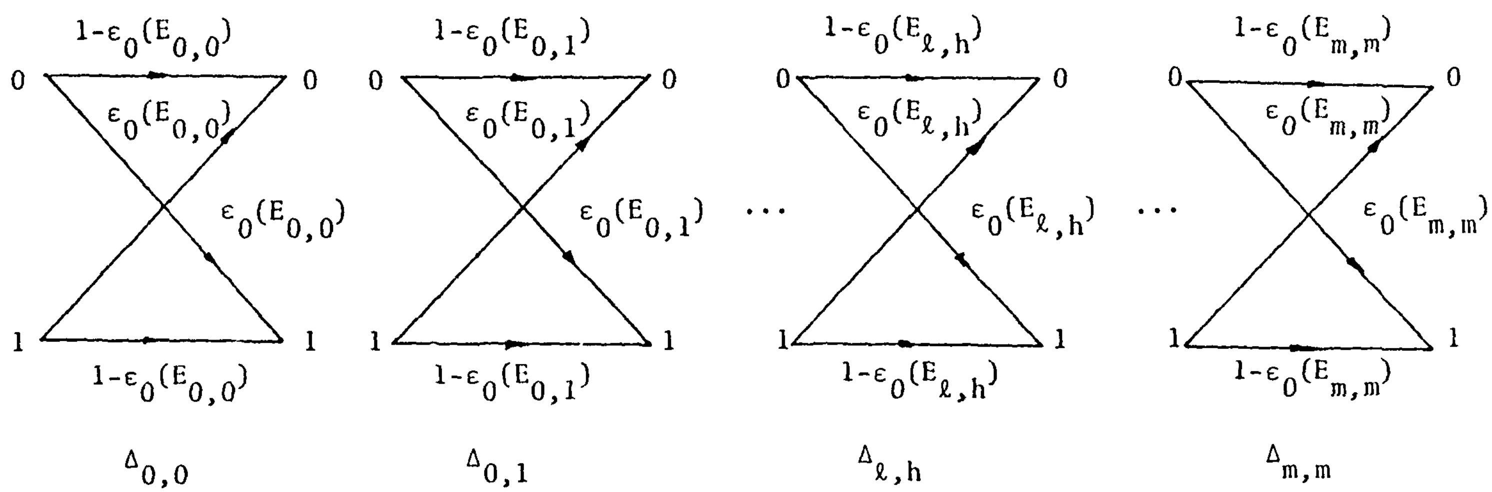

Figure 3.12. Outer Channel Resulting From Decoding Inner Code on a Burst-Noise Channel

3.6.3. Performance of the Concatenated Code on a Burst-Noise-Channel

If the n_b-bit block is transmitted over the component channel $\Delta_{\ell,h}$ of the outer channel, the probability of undetected error of the outer code is

$$P_{ud}^{(b)}(\varepsilon_0(E_{\ell,h})) = \sum_{i=d_b}^{n_b} A_i^{(b)}(\varepsilon_0(E_{\ell,h}))^i(1-\varepsilon_0(E_{\ell,h}))^{n_b-i}. \tag{3.61}$$

With the aid of the above outer channel model, the average probability of undetected error of the concatenated code is

$$P_{ud} = \sum_{h=0}^{m}\sum_{\ell=0}^{h} P_r\{E_{\ell,h}\}P_{ud}^{(b)}(\varepsilon_0(E_{\ell,h})). \tag{3.62}$$

For large m, the computation of (3.62) is very complex and time consuming. To reduce the computational work to a manageable amount, we seek an approximation to (3.62).

Define

$$\varepsilon_{max} = \max[\varepsilon_{a1/E}, \varepsilon_{a2/E}]. \tag{3.63}$$

It follows from (3.60) that

$$\varepsilon_0(E_{\ell,h}) \leq h\, \varepsilon_{max}/m \overset{\Delta}{=} \varepsilon_0(h), \tag{3.64}$$

and equality holds when ε_1 and ε_2 are equal, i.e., the inner channel is a memoryless BSC. Assuming that $P_{ud}^{(b)}(z)$ is an increasing function of z, $0 \leq z \leq 1/2$, we obtain from (3.62) and (3.64)

$$\begin{aligned} P_{ud} &\leq \sum_{h=0}^{m}\sum_{\ell=0}^{h} P_r\{E_{\ell,h}\}P_{ud}^{(b)}(\varepsilon_0(h)) \\ &= \sum_{h=0}^{m} P_{ud}^{(b)}(\varepsilon_0(h)) \sum_{\ell=0}^{h} P_r\{E_{\ell,h}\} = \sum_{h=0}^{m} P_{ud}^{(b)}(\varepsilon_0(h))\beta(h), \end{aligned} \tag{3.65}$$

where

$$\beta(h) = \sum_{\ell=0}^{h} P_r\{E_{\ell,h}\}, \quad 0 \leq h \leq m, \tag{3.66}$$

is the probability that h of the m decoded frames contain undetected errors (and the remaining m-h decoded frames are error free). $\beta(h)$ can be readily computed by a recursive method. To find $\beta(h)$, we model the decoded frame status as a Markov chain (see Figure 3.13). In state j, j = 1, 2, the decoded frame contains an undetected error with probability $P_{ud}^{(f)}(\varepsilon_j)$ and is error free with probability $P_c^{(f)}(\varepsilon_j)$.

Define $G(h,m) = P_r\{$h of the m decoded frames contain undetected errors / the inner channel starts in state 1$\}$ and $B(h,m) = P_r\{$h of the m decoded frames contain undetected errors / the inner channel starts in state 2$\}$. By applying a similar argument as in [40], we have

$$\beta(h) = \frac{p}{P+p} G(h,m) + \frac{P}{P+p} B(h,m), \quad 0 \leq h \leq m. \tag{3.67.1}$$

G(h,m) and B(h,m) can be found recursively from

$$G(h,m) = G(h,m-1)Q\, P_c^{(f)}(\varepsilon_1) + B(h,m-1)P\, P_c^{(f)}(\varepsilon_1)$$
$$+ G(h-1,m-1)Q\, P_{ud}^{(f)}(\varepsilon_1) + B(h-1,m-1)P\, P_{ud}^{(f)}(\varepsilon_1), \tag{3.67.2}$$

$$B(h,m) = B(h,m-1)q\, P_c^{(f)}(\varepsilon_2) + G(h,m-1)p\, P_c^{(f)}(\varepsilon_2)$$
$$+ B(h-1,m-1)q\, P_{ud}^{(f)}(\varepsilon_2) + G(h-1,m-1)p\, P_{ud}^{(f)}(\varepsilon_2), \tag{3.67.3}$$

$$G(0,1) = P_c^{(f)}(\varepsilon_1) \quad , \quad B(0,1) = P_c^{(f)}(\varepsilon_2),$$
$$G(1,1) = P_{ud}^{(f)}(\varepsilon_1) \quad , \quad B(1,1) = P_{ud}^{(f)}(\varepsilon_2) \tag{3.67.4}$$

we must also assign the values $G(h,m) = B(h,m) = 0$ when $h < 0$ or $h > m$.

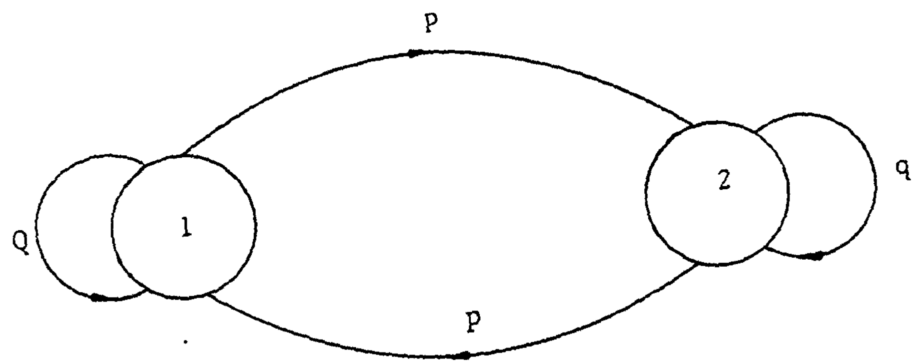

Figure 3.13. Decoded Frame Status

Note that if $\varepsilon_{a1/E} \approx \varepsilon_{a2/E}$, the upper bound of (3.65) is very close to (3.62). Fortunately, this is usually the case for $0 < \varepsilon_1 < \varepsilon_2 < 1/2$, especially for small ε_1 and ε_2, for then $\varepsilon_{a1/E} \approx d_f/n$.

Although in general the computation of (3.62) is very involved, in the following two important cases it can be handled quite easily.

1) $\varepsilon_1 = 0$. That is errors are not allowed to occur in state 1.Then (3.62) reduces to

$$P_{ud} = \sum_{h=0}^{m} P_r\{E_{h,h}\} P_{ud}^{(b)}(\varepsilon_0(E_{h,h})), \tag{3.68.1}$$

where

$$\varepsilon_0(E_{h,h}) = h\ \varepsilon_{a2/E} / m, \qquad 0 \le h \le m, \tag{3.68.2}$$

$$P_r(E_{h,h}) = \frac{p}{P+p} G(h,m) + \frac{P}{P+p} B(h,m), \quad 0 \le h \le m, \tag{3.68.3}$$

and both $G(h,m)$ and $B(h,m)$ can be found from (3.67.2)-(3.67.4) by letting $P_c^{(f)}(\varepsilon_1) = 1$ and $P_{ud}^{(f)}(\varepsilon_1) = 0$.

2) $P = 1-p$, i.e., the inner channel of Figure 3.11 becomes a BI channel, as shown in Figure 3.14. P_1 and P_2 are the probabilities of being in states 1 and 2, respectively. The probability $P_r\{E_{\ell,h}\}$ is given by

$$\begin{aligned} P_r\{E_{\ell,h}\} &= \sum_{s=\ell}^{m} \{\binom{s}{\ell}\binom{m-s}{h-\ell}[P_{ud}^{(f)}(\varepsilon_2)^{\ell}][P_c^{(f)}(\varepsilon_2)^{s-\ell}]P_2^{\,s} \\ &\quad \cdot [P_{ud}^{(f)}(\varepsilon_1)]^{h-\ell}[P_c^{(f)}(\varepsilon_1)]^{m-s-\ell}\ P_1^{\,m-s}\} \\ &\le [P_{ud}^{(f)}(\varepsilon_2]^{\ell}[P_{ud}^{(f)}(\varepsilon_1)]^{h-\ell}\sum_{s=\ell}^{m} \binom{s}{\ell}\binom{m-s}{h-\ell}\ P_2^{\,s}\ P_1^{\,m-s}, \\ &\qquad 0 \le \ell \le h \le m. \end{aligned} \tag{3.69}$$

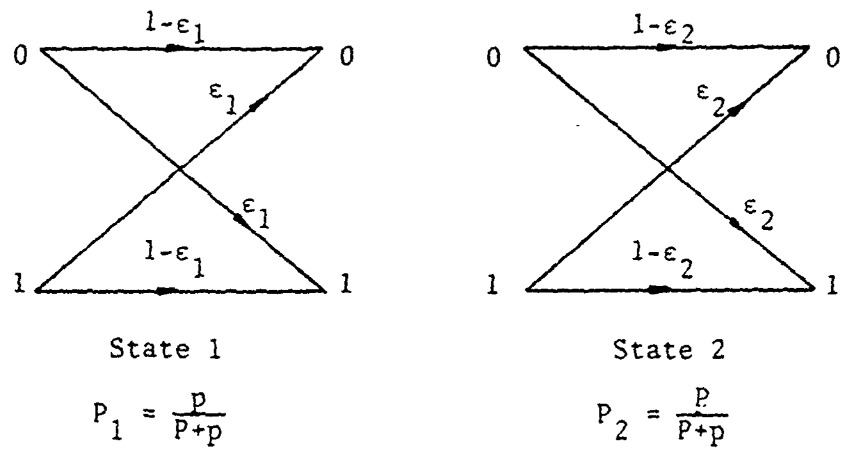

Figure 3.14. A BI Inner Channel

P_{ud} can be computed by using (3.69) in (3.62).

To evaluate the system throughput, again assume that selective-repeat ARQ is used. Let P_{ud}, P_r, and P_c denote the probabilities of an undetected error, of a block retransmission, and of correct decoding, respectively. Of course,

$$P_{ud} + P_r + P_c = 1. \tag{3.70}$$

In order to simplify the problem, we assume that retransmissions do not depend on the previous inner channel states. This is a reasonable assumption if the channel round-trip delay is large. Then the throughput of the system is [2]

$$\eta = \frac{k}{n}\frac{k_b}{n_b}(1-P_r) = \frac{k}{n}\frac{k_b}{n_b}(P_{ud}+P_c) \tag{3.71}$$

$$\simeq \frac{k}{n}\frac{k_b}{n_b}P_c , \tag{3.72}$$

and

$$P_c = \frac{p}{P+p}G(m) + \frac{P}{P+p}B(m), \tag{6.73.1}$$

$$G(m) = G(m-1)Q\, P_c^{(f)}(\varepsilon_1) + B(m-1)P\, P_c^{(f)}(\varepsilon_1), \tag{6.73.2}$$

$$B(m) = B(m-1)q\, P_c^{(f)}(\varepsilon_2) + G(m-1)p\, P_c^{(f)}(\varepsilon_2), \tag{6.73.3}$$

$$G(1) = P_c^{(f)}(\varepsilon_1) \quad , \quad B(1) = P_c^{(f)}(\varepsilon_2). \tag{6.73.4}$$

3.6.4. Examples on a Burst-Noise-Channel

Example 3.4

The same frame and outer codes are used as in Example 3.1. The

probability of undetected error, P_{ud}, and the system throughput, η, are plotted in Figures 3.15(a) and 3.15(b) for t=1, and in Figures 3.16(a) and 3.16(b) for t=0, respectively.

Example 3.5

The same coding scheme is used as in Example 3.3. P_{ud} and η are shown in Figures 3.17(a) and 3.17(b).

The performance of the concatenated coding scheme on burst-noise-channels heavily depends on the channel's parameters, especially on the high-to-low BER ratio, $\varepsilon_2/\varepsilon_1$. As shown in Figures 3.15(a)-3.17(b), for a given average BER $\bar{\varepsilon}$, with other parameters fixed, as the $\varepsilon_2/\varepsilon_1$ ratio becomes large, the system performance becomes poor. Our results indicate that on a burst-noise-channel for the same average BER, the system reliability degrades greatly, while the system throughput remains almost the same, compared with the same coding schemes on a memoryless BSC. For moderate values of average BER, high system reliability and throughput are still achievable using the concatenated coding system on burst-noise-channels.

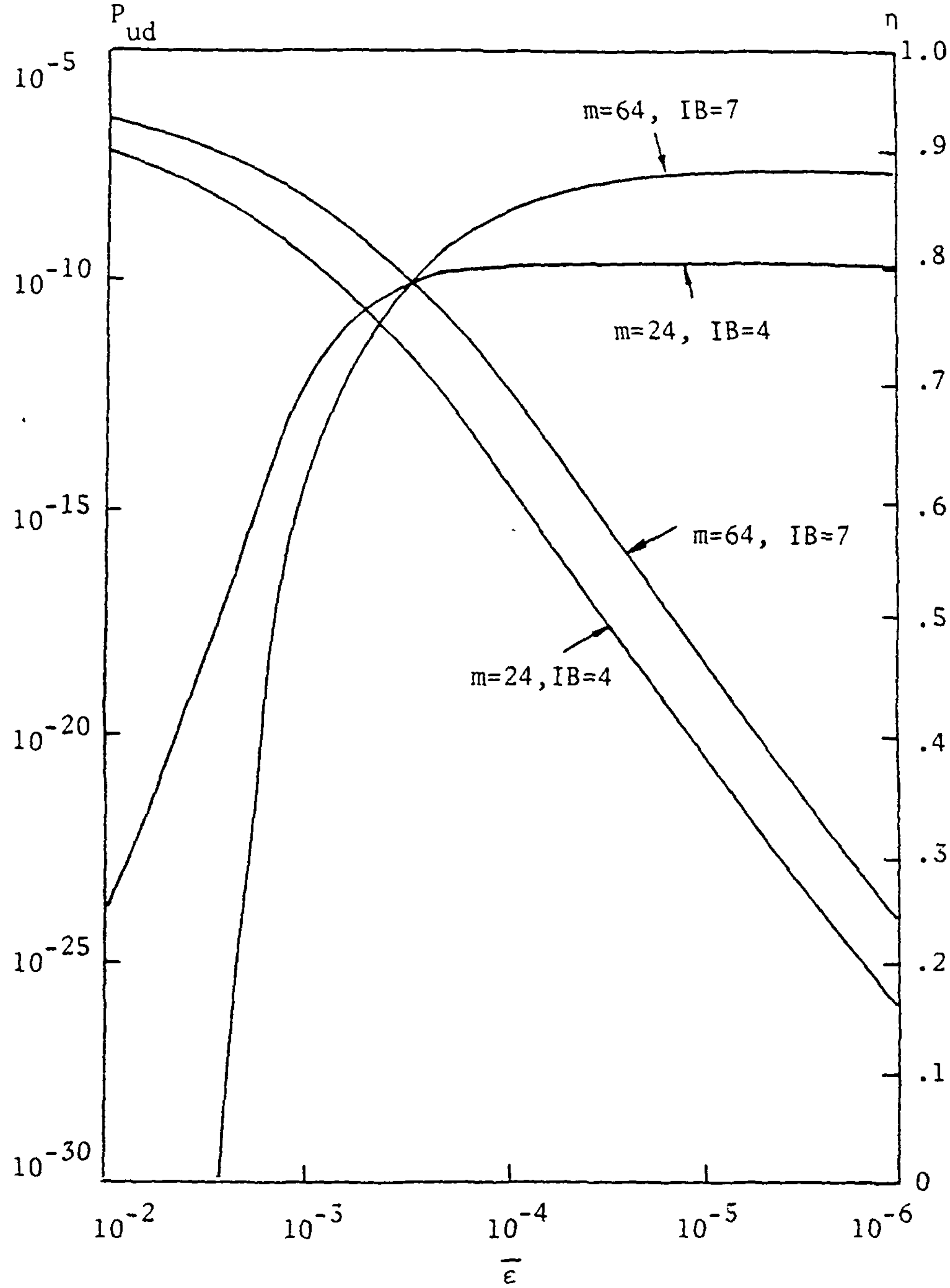

Figure 3.15(a). Performance of the Concatenated Code of Example 3.4 with $p_2 = 0.1$, $\overline{L} = 5$, $\varepsilon_2/\varepsilon_1 = 10$, $t = 1$

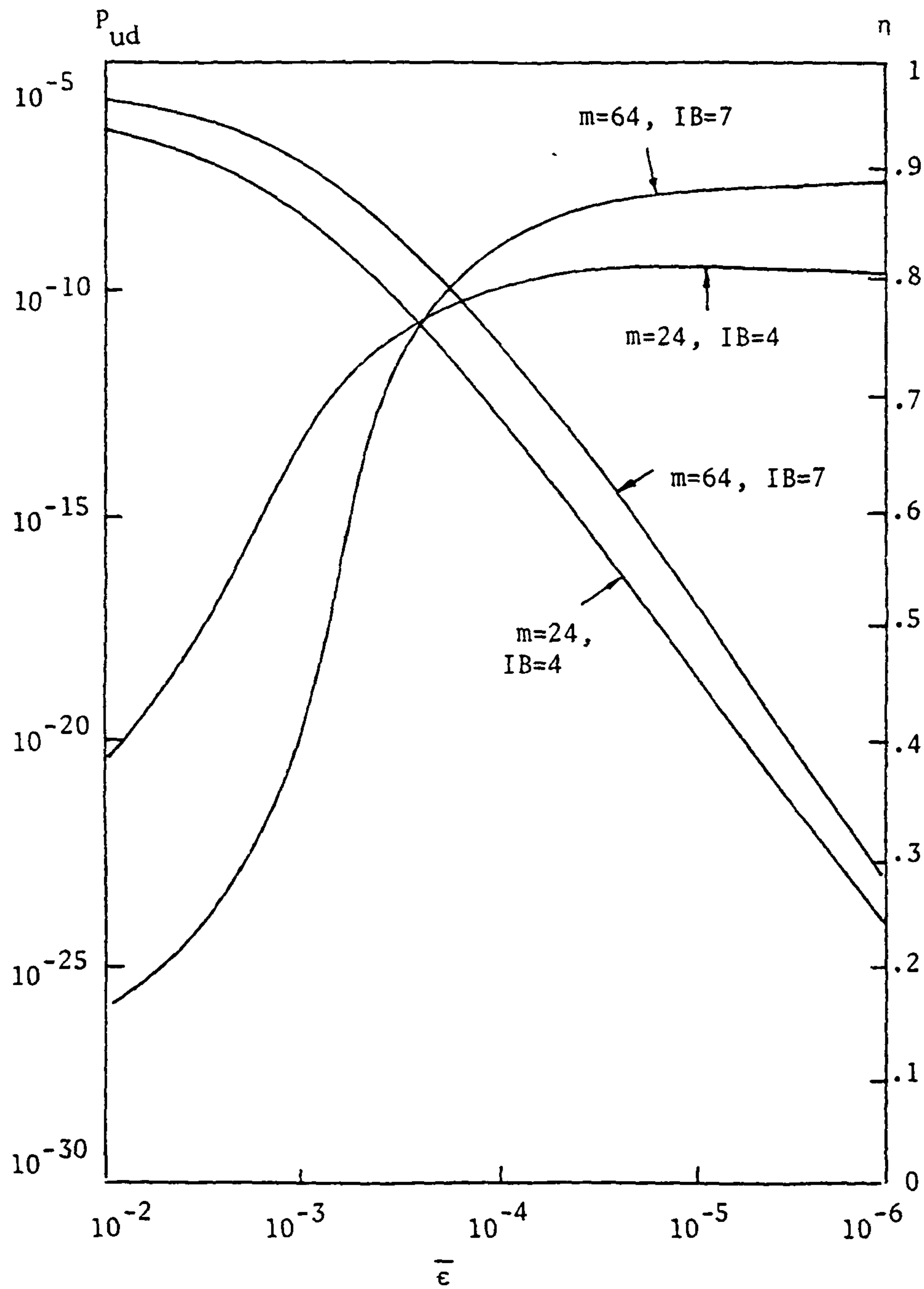

Figure 3.15(b). Performance of the Concatenated Code of Example 3.4 with p_2 = 0.1, $\bar{L}$=5, $\varepsilon_2/\varepsilon_1$ = 1000, t=1

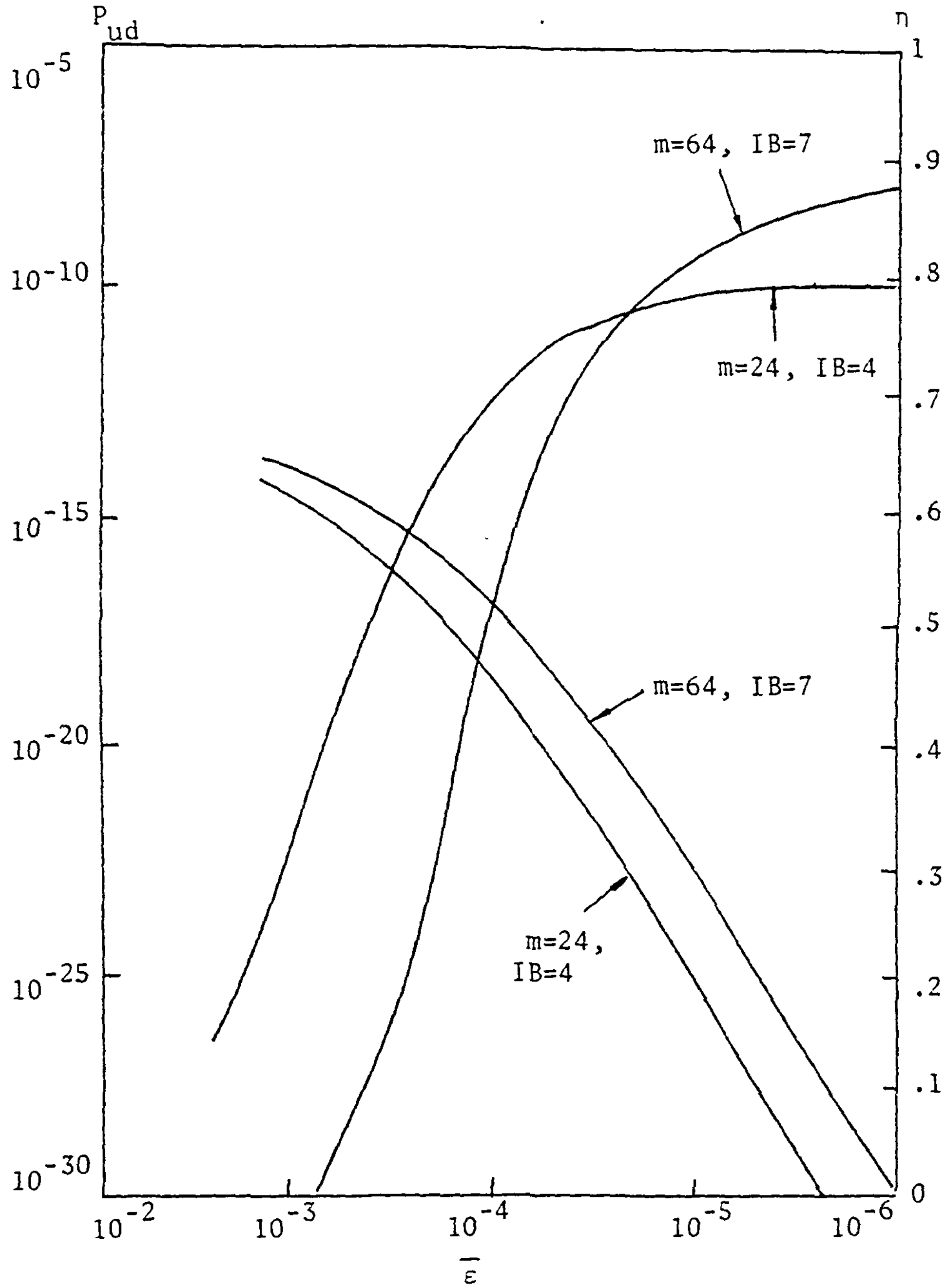

Figure 3.16(a). Performance of the Concatenated Code of Example 3.4 with $p_2 = 0.1$, $\overline{L}=5$, $\varepsilon_2/\varepsilon_1 = 10$, $t=0$

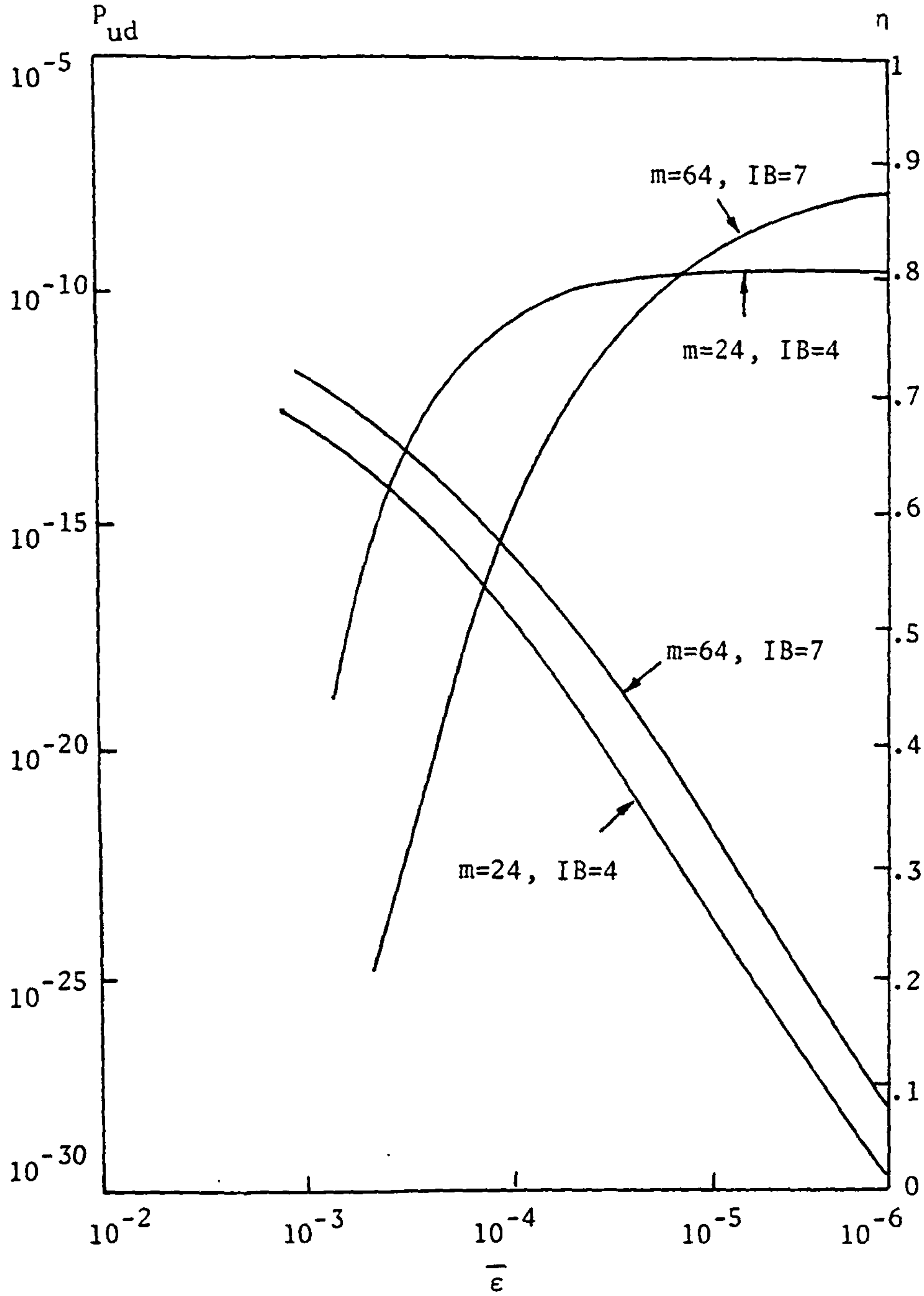

Figure 3.16(b). Performance of the Concatenated Code of Example 3.4 with $p_2 = 0.1$, $\overline{L}=5$, $\varepsilon_2/\varepsilon_1 = 1000$, $t=0$

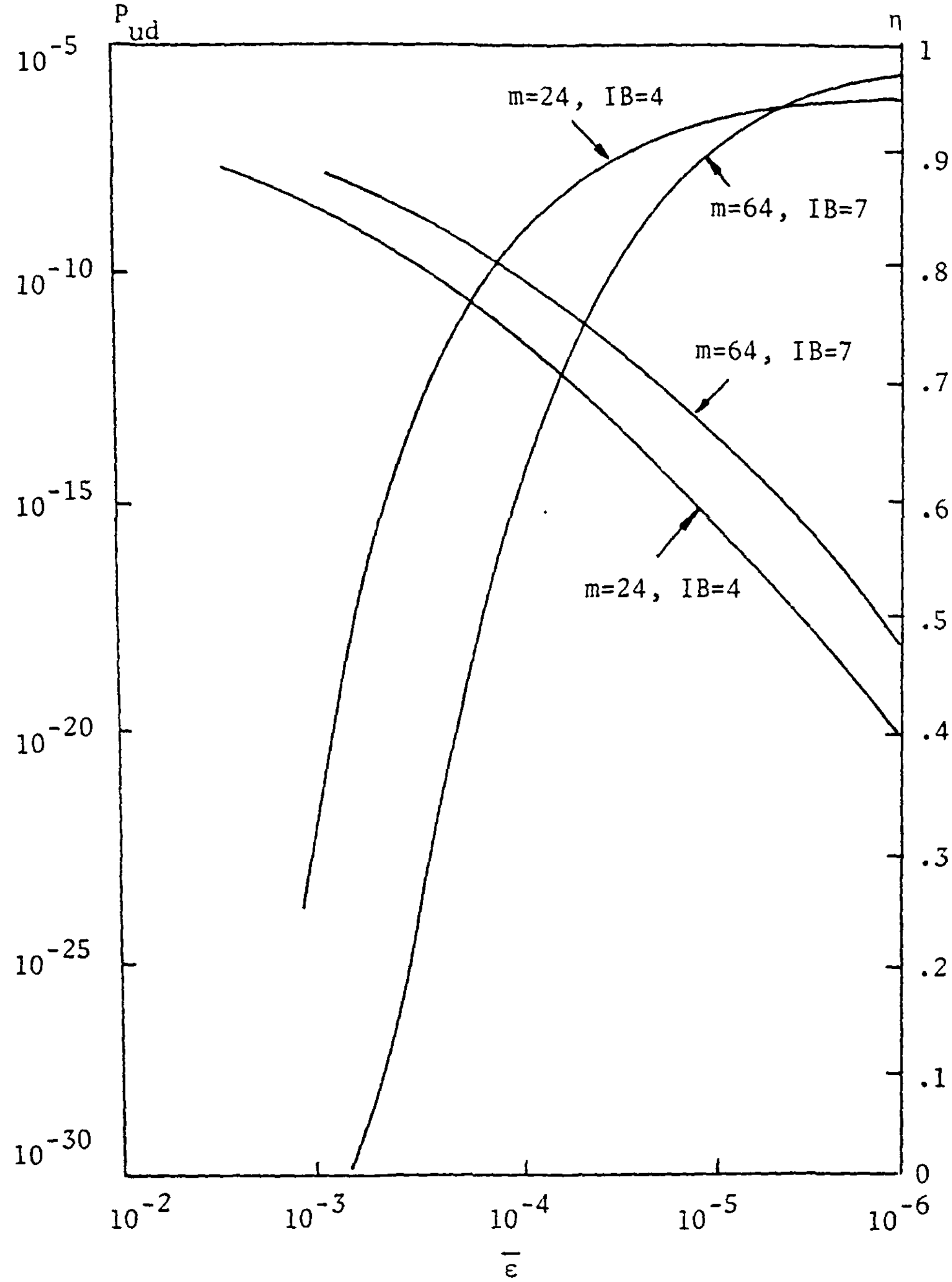

Figure 3.17(a). Performance of the Concatenated Code of Example 3.5 with p_2 = 0.1, L=5, $\varepsilon_2/\varepsilon_1$ = 10

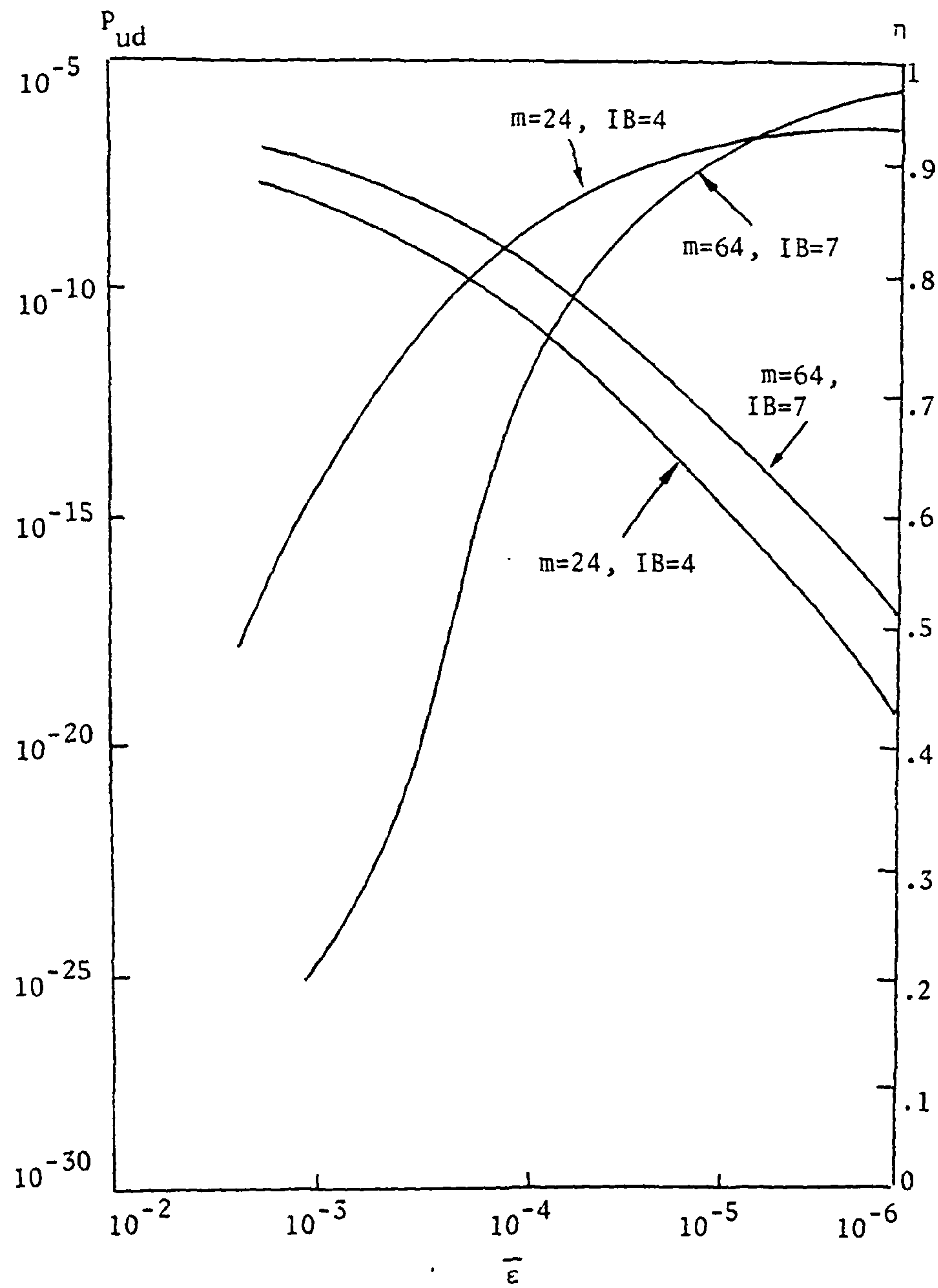

Figure 3.17(b). Performance of the Concatenated Code of Example 3.5 with $p_2 = 0.1$, L=5, $\varepsilon_2/\varepsilon_1 = 1000$

Appendix B

CHAPTER IV

ON THE OPTIMAL CODE RATE IN AN FEC SYSTEM

4.1. Problem Statement

A problem which frequently arises in forward-error-correction (FEC) coding is how to select a code from among various existing codes to obtain the best system performance. More specifically, what is the optimal code rate and code length that should be chosen?

Consider a discrete memoryless channel (DMC) with Q possible channel inputs $x = \{a_1, a_2, \cdots, a_Q\}$, which are specified by a distribution vector $\underline{q} = q\{(a_1), q(a_2), \cdots, q(a_Q)\}$, subject to the constraints

$$q(x) \geq 0 \quad , \quad \text{for every} \quad x \epsilon X,$$

and

$$\sum_x q(x) = 1. \tag{4.1}$$

The channel transition probabilities are denoted by $p(y|x)$ for every output $y \epsilon Y$ and input $x \epsilon X$. For such a channel, the Gallager function is defined by [22]

$$E_0(\rho, \underline{q}) = -\log_2 \sum_y [\sum_x q(x) p(y|x)^{\frac{1}{1+\rho}}]^{1+\rho} \quad , \quad 0 \leq \rho \leq 1, \tag{4.2}$$

and plays a central role in the channel coding theorem. The channel transition probabilities $p(y|x)$ in (4.2) depend on a parameter E_N/N_0, a nonnegative real number, called the channel symbol signal energy-to-noise power density ratio. When coding is used, if R is the code rate, as defined in Chapter II, then

$$E_N/N_0 = R \cdot E_b/N_0, \tag{4.3}$$

where E_b/N_0 is the information bit signal energy-to-noise power density ratio. For a given communication system, the average transmitter power is fixed, and so is E_b/N_0. Therefore, changing the code rate R varies the channel parameters. Most previous work on error control coding has not considered the influence of code rate on the transmission channel.

As a result, a communication engineer who must design a practical coded communication system may find little theoretical guidance in selecting the best code rate for the system. In this chapter we attempt to partially remedy this situation by finding the optimal code rate in an FEC system. By otpimal code rate, we mean the code rate which gives the smallest decoding error probability, or equivalently, the largest coding gain for a given E_b/N_0 and code length (for block codes) or constraint length (for convolutional codes). We begin by considering random coding error probability bounds [21,22] as a measure of system performance, and then proceed to consider random bounds on minimum distance. Both kinds of bounds will be used to study the optimal code rate problem.

4.2. The Optimal Block Code Rate

4.2.1. The Optimal Code Rate in Terms of the Error Probability Bound

The channel coding theorem says that the average decoding error probability over the ensemble of all block codes of length n and rate R for the Q-ary input DMC described in section 4.1 is bounded by [21,22]

$$\overline{P}_E < 2^{-nE_b(R)}, \tag{4.4}$$

where

$$E_b(R) = \max_{\underline{q}} \max_{0\leq\rho\leq 1} [E_0(\rho, \underline{q}) - \rho R], \tag{4.5}$$

and $E_0(\rho, \underline{q})$ is given by (4.1). It follows from (4.4) that at least one code in the ensemble must have P_E no greater than this ensemble bound.

Note that $E_0(\rho, \underline{q})$ is also a function of R. If we use the code length n as a measure of decoding complexity, from (4.4) and (4.5) we see that the code rate R should be chosen such that $E_b(R)$ is as large as possible. Formally, the optimization problem is

$$\begin{aligned} \text{maximize:} \quad & E_b(R) \\ \text{subject to:} \quad & E_b(R) > 0. \end{aligned} \tag{4.6}$$

The code rate R which satisfies (4.6) is defined as the optimal code rate and is denoted by $R_{opt}^{(b)}$.

Another important quantity in describing the performance of coded communication systems is the <u>computational cutoff rate</u> R_0. It is defined as the largest number for which there is a bound with a linear exponent, i.e., a bound of the form [21,22]

$$\overline{P}_E < 2^{-n(R_0 - R)}, \tag{4.7}$$

on the average decoding error probability of all codes of length n and rate R on the Q-ary DMC described in section 4.1, where

$$\begin{aligned} R_0 &= \max_{\underline{q}} E_0(1, \underline{q}) \\ &= \max_{\underline{q}} \{-\log_2 \sum_y [\sum_x q(x) \sqrt{p(y|x)}\ ]^2\}. \end{aligned} \tag{4.8}$$

If we define

$$E_{b0}(R) = R_0 - R, \tag{4.9}$$

the optimization problem is

$$\begin{aligned} \text{maximize:} \quad & E_{b0}(R) \\ \text{subject to:} \quad & E_{b0}(R) > 0, \end{aligned} \tag{4.10}$$

and the code rate R which satisfies (4.10) is denoted by $R_{opt}^{(b0)}$.

It can be seen from (4.1), (4.5), and (4.8) that as long as $p(y|x)$ is continuous in R, both $E_b(R)$ and $E_{b0}(R)$ are continuous functions of R over a closed region. Therefore, there exists at least one value of R which satisfies (4.6), and at least one value of R which satisfies (4.10). More general statements about (4.6) and (4.10) cannot be made because the channel transition probabilities $p(y|x)$ depend on R, E_b/N_0, the modulation/demodulation scheme used, the channel noise characteristics, the channel output quantization method, etc. This fact is made more clear by the following examples.

Example 4.1

Suppose that BPSK or BFSK modulation is used over an AWGN channel. If the demodulator makes hard quantization, a BSC results with bit error rate $\varepsilon = \varepsilon(R \cdot E_b/N_0)$, where ε is a decreasing convex U function of R, as shown in Appendix A. From (4.1) and (4.5) we obtain

$$\begin{aligned} E_b(R) &= \rho_0 - (1+\rho_0)\log_2[\varepsilon^{\frac{1}{1+\rho_0}} + (1-\varepsilon)^{\frac{1}{1+\rho_0}}] - \rho_0 R \\ &= E_0(\rho_0) - \rho_0 R, \end{aligned} \tag{4.11}$$

where

$$E_0(\rho_0) = \rho_0 - (1+\rho_0)\log_2[\varepsilon^{\frac{1}{1+\rho_0}} + (1-\varepsilon)^{\frac{1}{1+\rho_0}}],$$

and ρ_0 is the value of $\rho \epsilon [0, 1]$ which maximizes $E_b(R)$. It is easy to show that

$$\frac{dE_0(\rho_0)}{d\varepsilon} < 0 \quad , \quad 0 < \varepsilon < \frac{1}{2} , \tag{4.12.1}$$

and

$$\frac{d^2E_0(\rho_0)}{d\varepsilon^2} > 0 \quad , \quad 0 < \varepsilon < \frac{1}{2} . \tag{4.12.2}$$

We also have

$$\frac{dE_b(R)}{dR} = \frac{dE_0(\rho_0)}{d\varepsilon} \frac{d\varepsilon}{dR} - \rho_0 , \tag{4.13.1}$$

and

$$\frac{d^2E_b(R)}{dR^2} = \frac{d^2E_0(\rho_0)}{d\varepsilon^2} \left(\frac{d\varepsilon}{dR} \right)^2 + \frac{dE_0(\rho_0)}{d\varepsilon} \frac{d^2\varepsilon}{dR^2} . \tag{4.13.2}$$

Since $= (R \cdot E_b/N_0)$ is a decreasing convex U function of R, it follows that

$$\frac{d\varepsilon}{dR} < 0 \quad , \quad R \geq 0 , \tag{4.14.1}$$

$$\frac{d^2\varepsilon}{dR^2} > 0 \quad , \quad R \leq 0 . \tag{4.14.2}$$

Now let C(R) denote the channel capacity, which is a function of the code rate R. From (4.12.1)-(4.14.2) we can draw the following conclusions:

1) $$\frac{d^2E_b(R)}{dR^2} \leq 0, \text{ if } \frac{d^2E_0(\rho_0)}{d\varepsilon^2} \left(\frac{d\varepsilon}{dR} \right)^2 \leq \left| \frac{dE_0(\rho)}{d\varepsilon} \frac{d^2\varepsilon}{dR^2} \right|$$

for $0 \leq R < C(R)$,

and the optimal code rate $R_{opt}^{(b)}$ is the unique solution of the following equation

$$\frac{dE_0(\rho_0)}{d\varepsilon} \cdot \frac{d\varepsilon}{dR} = \rho_0; \tag{4.15}$$

2) $$\frac{d^2E_b(R)}{dR^2} > 0, \text{ if } \frac{d^2E_0(\rho_0)}{d\varepsilon^2} \cdot \left(\frac{d\varepsilon}{dR}\right)^2 > \left| \frac{dE_0(\rho_0)}{d\varepsilon} \frac{d^2\varepsilon}{dR^2} \right|,$$

for $0 \le R \le C(R)$,

and

$$R_{opt}^{(b)} = R' , \tag{4.16}$$

where R' maximizes $C(R')$;

3) If $E_b(R)$ is not a convex function for $0 \le R \le C(R)$, $R_{opt}^{(b)}$ must be found numerically.

By replacing ρ_0 with 1 and $C(R)$ with R_0, respectively, similar conclusions apply to the function $E_{b0}(R)$.

This example is used to indicate that even for the particularly simple, often studied, BSC, finding the optimal code rate $R_{opt}^{(b)}$ is rather complex. The computation of $R_{opt}^{(b)}$ for explicit channels is generally very involved, because the function $E_b(R)$ (or $E_{b0}(R)$) depends on R, E_b/N_0, the modulation/demodulation scheme used, the channel noise characteristics, the channel output quantization method, etc. Fortunately, direct numerical computation can be carried out easily by computer. The optimal code rate $R_{opt}^{(b)}$ for BPSK modulation over an AWGN channel with demodulator output hard quantization is given in Table 4.1 and depicted in Figure 4.1 as a function of E_b/N_0.

Table 4.1. $R_{opt}^{(b)}$ for BPSK Modulation over an AWGN Channel with Output Hard Quantization

E_b/N_0	$R_{opt}^{(b)}$	C	$E_b(R_{opt}^{(b)})$
1.1 (T)	0	0	0
1.2	0.06	0.064	0.0001
1.5	0.17	0.209	0.0024
2.0	0.27	0.392	0.0121
2.5	0.32	0.522	0.0388
3.0	0.34	0.610	0.0691
4.0	0.37	0.746	0.1429
5.0	0.38	0.828	0.2238
6.0	0.38	0.879	0.2938
7.0	0.37	0.910	0.3520
8.0	0.36	0.931	0.4010
9.0	0.34	0.942	0.4420
10.0	0.33	0.954	0.4780

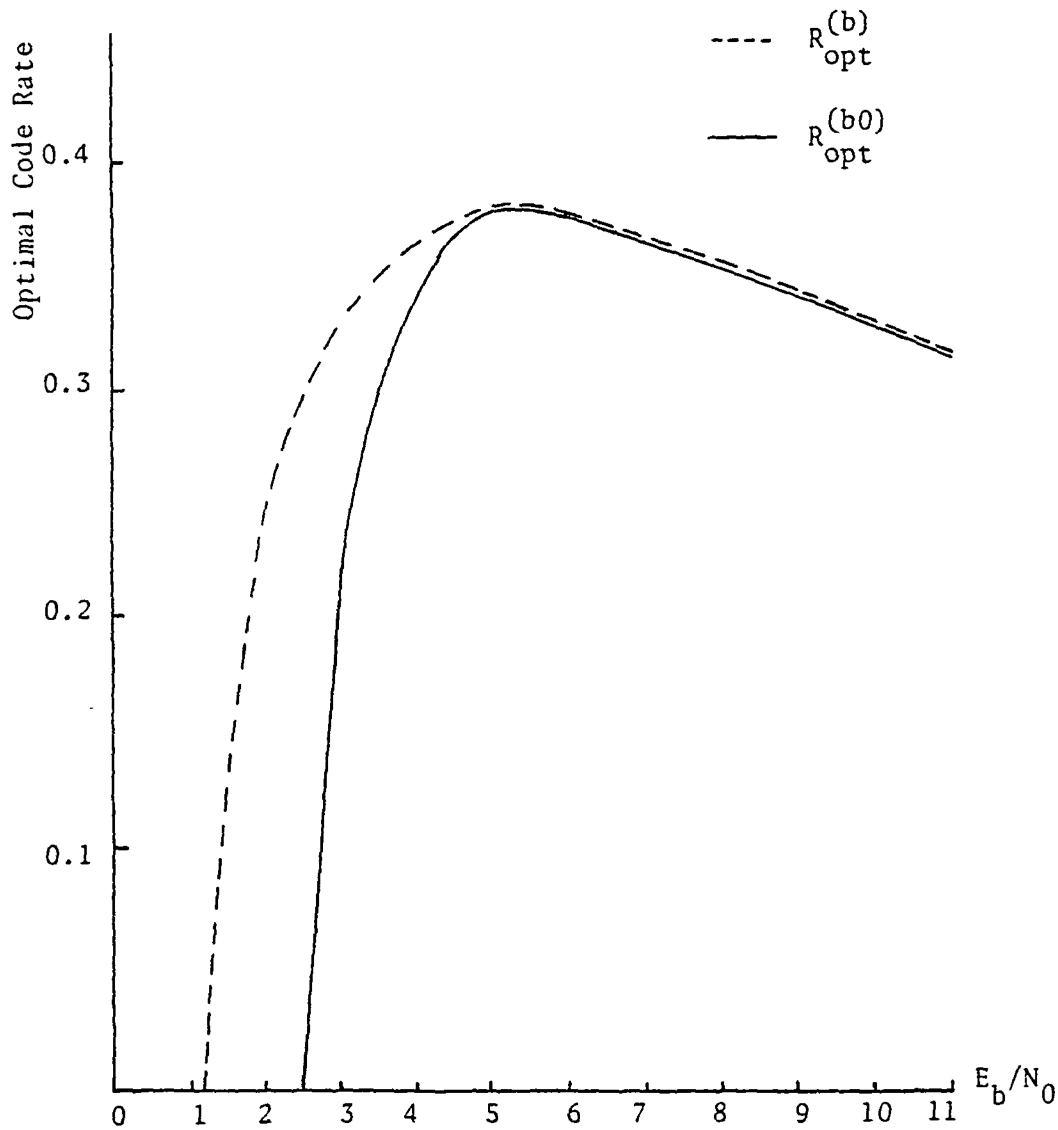

Figure 4.1. Optimal Block Code Rate for BPSK Modulation over an AWGN Channel with Output Hard Quantization

Example 4.2

For M-ary PSK modulation over an AWGN channel with no demodulator output quantization, the computational cutoff rate is given by [23]

$$R_0 = -\log_2 \frac{1}{M} \sum_{k=1}^{M} \exp[-R \frac{E_b}{N_0} \sin^2 \frac{k\pi}{M}], \tag{4.17}$$

where M is a power of 2. It can be shown that [see Appendix B]

$$\frac{dR_0}{dR} \geq 0 \quad , \quad \text{for} \quad E_b/N_0 \geq 0 \quad , \quad R \geq 0, \tag{4.18}$$

and

$$\frac{d^2R}{dR^2} \leq 0 \quad , \quad \text{for} \quad E_b/N_0 \geq 0 \quad , \quad R \geq 0. \tag{4.19}$$

Equality in both (4.18) and (4.19) holds only when $E_b/N_0 = 0$. It follows from (4.19) that

$$\frac{d^2E_{b0}(R)}{dR^2} \leq 0 \quad , \quad \text{for} \quad E_b/N_0 \geq 0 \quad , \quad R \geq 0. \tag{4.20}$$

Therefore, $E_{b0}(R) = R_0 - R$ is a strict convex $\cap$ function of $R \geq 0$ for $E_b/N_0 > 0$. The equation for a stationary point of $E_{b0}(R)$ with respect to R is

$$\frac{dR_0}{dR} - 1 = 0. \tag{4.21}$$

Since $d^2R_0/dR^2 \leq 0$, any solution of (4.21) in the range $0 \leq R \leq 1$, subject to $E_{b0}(R) \geq 0$, maximizes $E_{b0}(R)$, and hence gives the optimal code rate $R_{opt}^{(b0)}$.

If M=2 and 4, i.e., BPSK and QPSK modulation, the optimal code rate can be found explicitly by solving (4.21) and is given by

$$R_{opt}^{(b0)} = \begin{cases} -\frac{1}{E_b/N_0} \ln\left[\frac{\ln 2}{E_b/N_0 - \ln 2}\right], & \text{for BPSK, } E_b/N_0 \geq 2\ln 2, \\ -\frac{2}{E_b/N_0} \ln\left[\frac{\ln 2}{E_b/N_0 - \ln 2}\right], & \text{for QPSK, } E_b/N_0 \geq 2\ln 2. \end{cases} \tag{4.22}$$

For $E_b/N_0 < 2\ln 2$, we always have $E_{b0}(R) < 0$, so with BPSK (or QPSK) modulation on an AWGN channel, any block coding technique will require an E_b/N_0 of greater than $10 \log_{10}(2\ln 2) = 1.42$ dB for small error rates and reasonable implementation complexity, regardless of the code rate or of how many quantization levels are used at the demodulator output, as stated in [41]. Therefore, we call $E_b/N_0 = 2\ln 2$ the "information bit signal energy-to-noise power density ratio threshold", and denote it by T. The optimal code rates as a function of E_b/N_0 for this example are listed in Tables 4.2(a)-4.2(d) and plotted in Figure 4.2.

The optimal code rates $R_{opt}^{(b0)}$, for BPSK modulation over an AWGN channel with demodulator output hard quantization and for MFSK modulation over an AWGN channel with demodulator output hard quantization are listed in Tables 4.3 and 4.4 and depicted in Figures 4.1 and 4.3, respectively. In the tables we also indicate the E_b/N_0 threshold, T, for each case.

From these examples we observe that the value of $R_{opt}^{(b)}$ (or $R_{opt}^{(b0)}$) is relatively small and is inversely proportional to E_b/N_0 for relatively large values of E_b/N_0. For small values of $R_{opt}^{(b)}$ (or $R_{opt}^{(b0)}$), $E_N/N_0 = R_{opt}^{(b)} \cdot E_b/N_0$ (or $= R_{opt}^{(b0)} \cdot E_b/N_0$), and the "channel symbol signal energy-to-noise power density ratio" becomes small compared with E_b/N_0, and consequently results in a "noisy" channel, or a "higher" channel bit error rate. The interesting fact is to note that the "high" bit error rate is

Table 4.2(a). $R_{opt}^{(b0)}$ for BPSK Modulation over an AWGN Channel with no Output Quantization

E_b/N_0	$R_{opt}^{(b0)}$	R_0	$E_{b0}(R_{opt}^{(b0)})$
1.39 (T)	0	0	0
2.0	0.317	0.386	0.069
3.0	0.401	0.621	0.220
3.15	0.402	0.635	0.233
4.0	0.391	0.726	0.335
5.0	0.365	0.784	0.419
6.0	0.339	0.823	0.484
7.0	0.315	0.849	0.534
8.0	0.294	0.869	0.575
9.0	0.276	0.884	0.608
10.0	0.260	0.897	0.637
11.0	0.245	0.906	0.661
12.0	0.233	0.915	0.682
13.0	0.221	0.921	0.700
14.0	0.211	0.927	0.716
15.0	0.202	0.932	0.730

Table 4.2(b). $R_{opt}^{(b0)}$ for QPSK Modulation over an AWGN Channel with no Output Quantization

E_b/N_0	$R_{opt}^{(b0)}$	R_0	$E_{b0}(R_{opt}^{(b0)})$
1.39 (T)	0	0	0
2.0	0.634	0.772	0.138
3.0	0.802	1.242	0.440
3.15	0.804	1.270	0.466
4.0	0.782	1.452	0.670
5.0	0.730	1.568	0.838
6.0	0.678	1.646	0.968
7.0	0.630	1.698	1.068
8.0	0.588	1.738	1.150
9.0	0.552	1.768	1.216
10.0	0.520	1.794	1.274
11.0	0.490	1.812	1.322
12.0	0.466	1.830	1.364
13.0	0.442	1.842	1.400
14.0	0.422	1.854	1.432
15.0	0.404	1.864	1.460

Table 4.2(c). $R_{opt}^{(b0)}$ for 8-ary PSK Modulation over an AWGN Channel with no Output Quantization

E_b/N_0	$R_{opt}^{(b0)}$	R_0	$E_{b0}(R_{opt}^{(b0)})$
1.39 (T)	0	0	0
2.0	0.645	0.783	0.138
3.0	0.855	1.308	0.453
4.0	0.879	1.584	0.705
5.0	0.864	1.764	0.900
6.0	0.846	1.902	1.056
7.0	0.825	2.010	1.185
8.0	0.807	2.100	1.293
9.0	0.792	2.178	1.386
10.0	0.780	2.250	1.470
11.0	0.765	2.310	1.545
12.0	0.753	2.364	1.611
13.0	0.741	2.412	1.671
14.0	0.726	2.451	1.725
15.0	0.714	2.487	1.773

Table 4.2(d). $R_{opt}^{(b0)}$ for 16-ary PSK Modulation over an AWGN Channel with no Output Quantization

E_b/N_0	$R_{opt}^{(b0)}$	R_0	$E_{b0}(R_{opt}^{(b0)})$
1.39 (T)	0	0	0
2.0	0.64	0.780	0.140
3.0	0.84	1.292	0.452
4.0	0.88	1.584	0.704
5.0	0.88	1.780	0.900
6.0	0.84	1.896	1.056
7.0	0.84	2.024	1.184
8.0	0.80	2.092	1.292
9.0	0.80	2.188	1.388
10.0	0.80	2.272	1.472
11.0	0.80	2.348	1.548
12.0	0.76	2.376	1.616
13.0	0.76	2.436	1.676
14.0	0.76	2.492	1.732
15.0	0.76	2.548	1.788

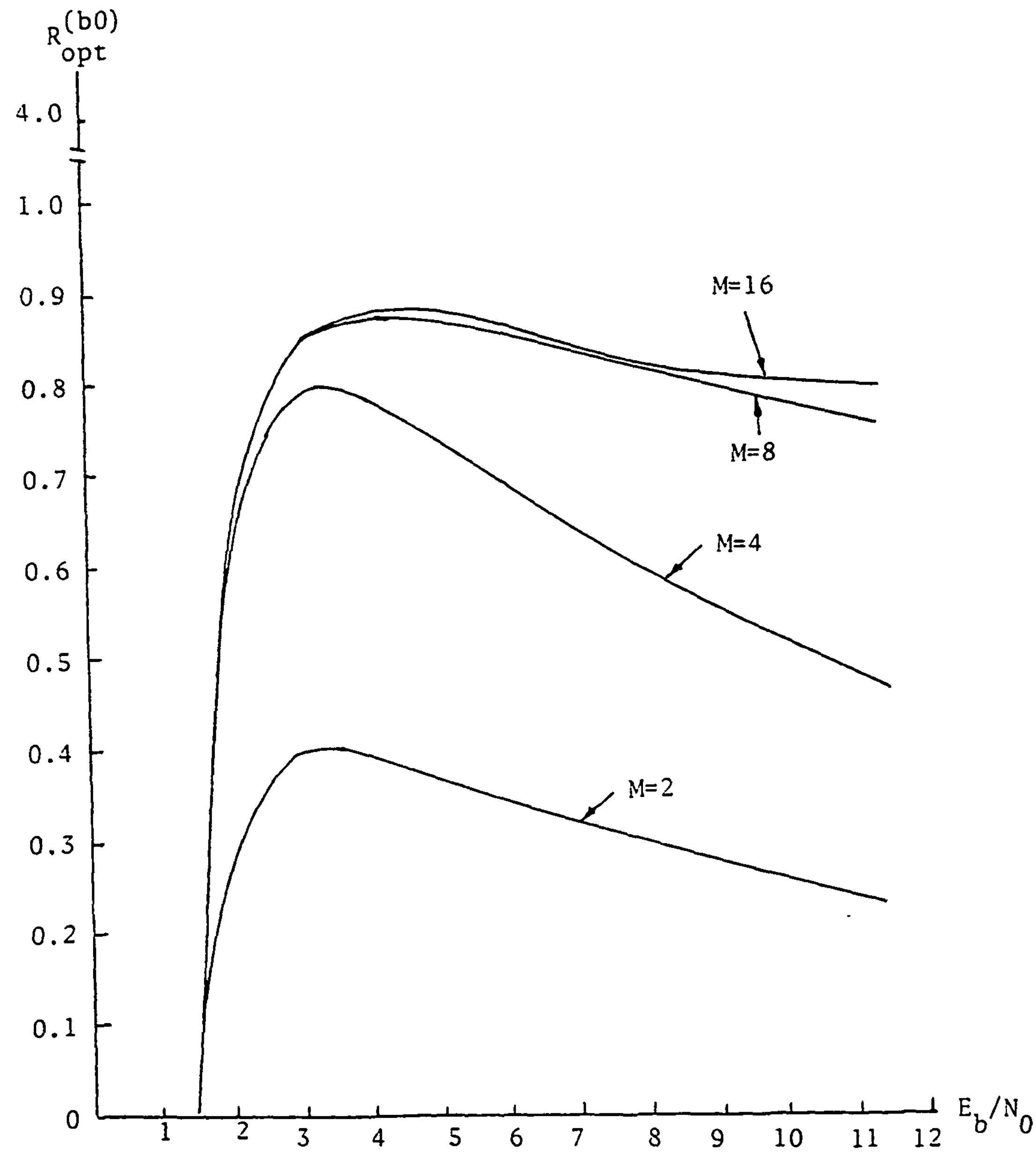

Figure 4.2. Optimal Block Code Rate for MPSK Modulation over an AWGN Channel with no Output Quantization

Table 4.3. $R_{opt}^{(b0)}$ for BPSK Modulation over an AWGN Channel with Output Hard Quantization

E_b/N_0	$R_{opt}^{(b0)}$	R_0	$E_{b0}(R_{opt}^{(b0)})$
2.1 (T)	0	0	0
3.0	0.26	0.308	0.048
4.0	0.36	0.500	0.140
5.0	0.38	0.604	0.224
6.0	0.38	0.674	0.294
7.0	0.37	0.722	0.352
8.0	0.36	0.761	0.401
9.0	0.34	0.782	0.442
10.0	0.33	0.808	0.478
11.0	0.32	0.829	0.509
12.0	0.31	0.846	0.536
13.0	0.29	0.850	0.560
14.0	0.28	0.861	0.581
15.0	0.27	0.870	0.600

Table 4.4(a). $R_{opt}^{(b0)}$ for BFSK Modulation over an AWGN Channel with Output Hard Quantization

E_b/N_0	$R_{opt}^{(b0)}$	R_0	$E_{b0}(R_{opt}^{(b0)})$
9.5 (T)	0	0	0
10.0	0.49	0.508	0.018
12.0	0.50	0.609	0.109
15.0	0.48	0.699	0.219
20.0	0.43	0.781	0.351
25.0	0.39	0.832	0.442
30.0	0.35	0.859	0.509
35.0	0.32	0.881	0.561
40.0	0.29	0.892	0.602
45.0	0.27	0.905	0.635
50.0	0.25	0.913	0.663
55.0	0.24	0.927	0.687
60.0	0.22	0.927	0.707
65.0	0.21	0.934	0.724
70.0	0.20	0.940	0.720
75.0	0.19	0.943	0.753
80.0	0.18	0.945	0.765

Table 4.4(b). $R_{opt}^{(b0)}$ for QFSK Modulation over an AWGN Channel with Output Hard Quantization

E_b/N_0	$R_{opt}^{(b0)}$	R_0	$E_{b0}(R_{opt}^{(b0)})$
5.5 (T)	0	0	0
6.0	1.04	1.088	0.048
7.0	1.06	1.272	0.212
8.0	1.04	1.392	0.352
8.5	1.02	1.434	0.414
10.0	0.96	1.534	0.574
15.0	0.78	1.708	0.928
20.0	0.66	1.792	1.132
25.0	0.56	1.828	1.268
30.0	0.50	1.864	1.364
35.0	0.44	1.876	1.436
40.0	0.40	1.894	1.494
45.0	0.36	1.898	1.538
50.0	0.34	1.916	1.576
55.0	0.32	1.928	1.608
60.0	0.30	1.934	1.634
65.0	0.28	1.938	1.658
70.0	0.26	1.938	1.678
80.0	0.24	1.950	1.710

Table 4.4(c). $R_{opt}^{(b0)}$ for 8-ary FSK Modulation over an AWGN Channel with Output Hard Quantization

E_b/N_0	$R_{opt}^{(b0)}$	R_0	$E_{b0}(R_{opt}^{(b0)})$
4.5 (T)	0	0	0
5.0	1.71	1.881	0.171
6.0	1.65	2.130	0.480
8.0	1.47	2.400	0.930
10.0	1.29	2.529	1.239
13.0	1.11	2.664	1.554
15.0	0.99	2.694	1.704
18.0	0.87	2.745	1.875
20.0	0.81	2.775	1.965
25.0	0.69	2.823	2.133
30.0	0.60	2.853	2.253
35.0	0.54	2.880	2.340
40.0	0.48	2.889	2.409
45.0	0.45	2.913	2.463
50.0	0.42	2.928	2.508
55.0	0.39	2.934	2.544
60.0	0.36	2.937	2.577
65.0	0.33	2.937	2.604
70.0	0.30	2.928	2.628
80.0	0.27	2.937	2.667

Table 4.4(d). $R_{opt}^{(b0)}$ for 16-ary FSK Modulation over an AWGN Channel with Output Hard Quantization

E_b/N_0	$R_{opt}^{(b0)}$	R_0	$E_{b0}(R_{opt}^{(b0)})$
3.5 (T)	0	0	0
4.0	2.44	2.464	0.024
5.0	2.36	2.924	0.564
6.0	2.16	3.136	0.976
8.0	1.84	3.392	1.552
10.0	1.60	3.536	1.936
13.0	1.32	3.640	2.320
15.0	1.20	3.704	2.504
20.0	0.96	3.772	2.812
25.0	0.80	3.812	3.012
30.0	0.72	3.872	3.152
35.0	0.64	3.892	3.252
40.0	0.56	3.892	3.332
45.0	0.52	3.916	3.396
50.0	0.48	3.928	3.448
55.0	0.44	3.932	3.492
60.0	0.40	3.928	3.528
65.0	0.36	3.916	3.556
75.0	0.32	3.928	3.608

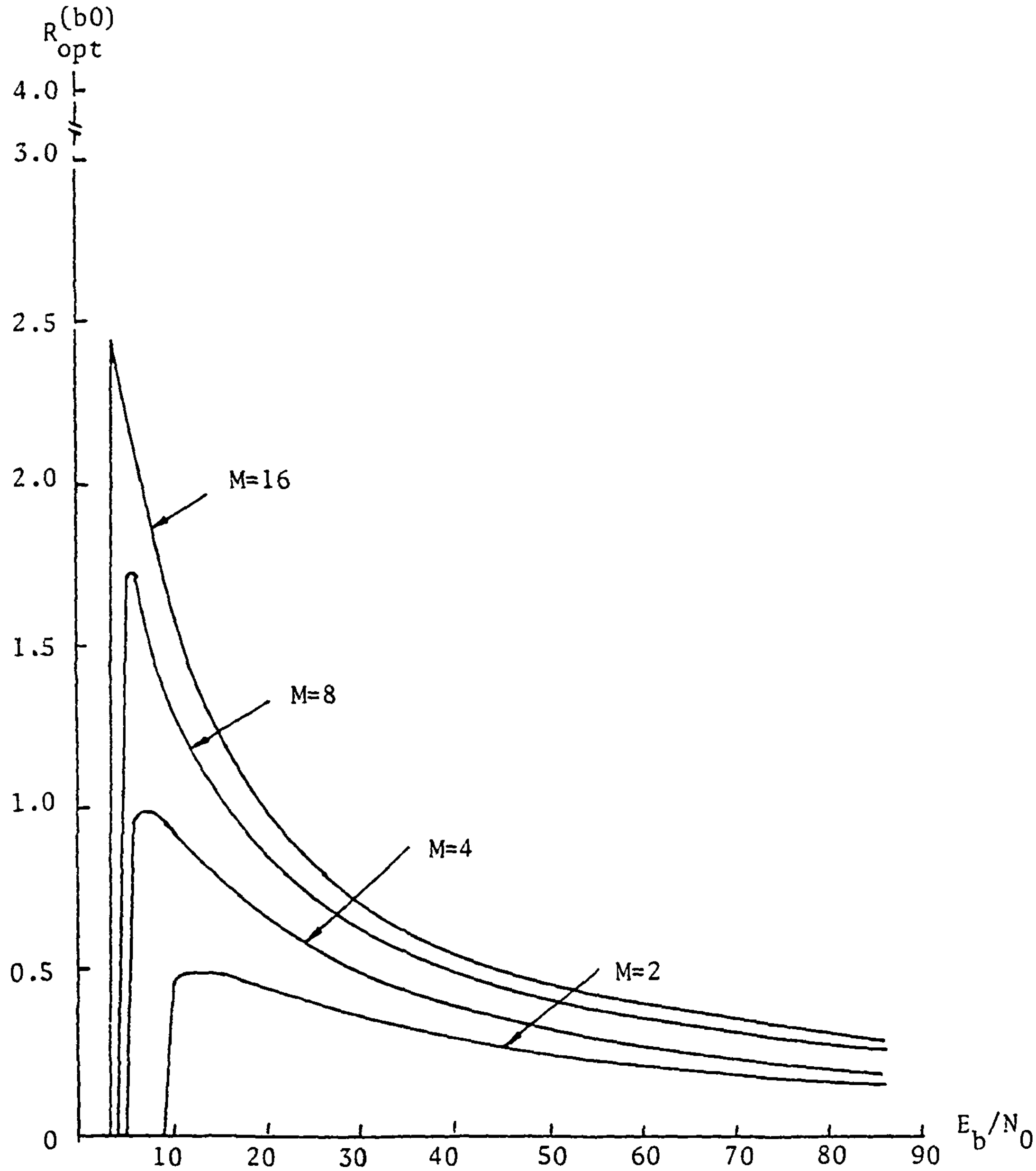

Figure 4.3. Optimal Block Code Rate for MFSK Modulation over an AWGN Channel with Output Hard Quantization

compensated for by the powerful error correcting capability of the codes of rate $R_{opt}^{(b)}$ (or $R_{opt}^{(b0)}$), and the net effect is that we obtain the lowest decoding error probability, or equivalently, the largest coding gain. The trade off is in a reduction of the system throughput. However, a relatively small optimal code rate need not reduce throughput if additional bandwidth is available on the channel. Satellite and deep space channels, in particular, are not nearly as much bandwidth limited as they are power limited.

In the practically interesting range of values for E_b/N_0, the values of $R_{opt}^{(b)}$ (or $R_{opt}^{(b0)}$) remain quite stable. For example, in Figure 4.1, for BPSK modulation with output hard quantization, when $E_b/N_0 \in [5,9)$, which approximately corresponds to a channel bit error rate $\varepsilon = 10^{-3} \sim 10^{-5}$, the optimal code rate $R_{opt}^{(b)} = 0.321 \pm 0.045$. This fairly stable optimum code rate shows that optimum system performance is not overly sensitive to the channel noise.

Note that $E_b(R) \geq 0$ for $0 \leq R \leq C$, while $E_{b0}(R) \geq 0$ only for $0 \leq R \leq R_0$ [21,22]. The channel capacity C is the absolute upper limit on the rate of a code. The channel cutoff rate R_0 is the upper limit for practically implementable systems [42]. Following the same line of reasoning, we believe that $R_{opt}^{(b)}$ is the upper limit on the optimal code rate, while $R_{opt}^{(b0)}$ more closely matches the real optimal code rate. For BPSK modulation on an AWGN channel with demodulator output hard quantization, Figure 4.1 shows the difference between $R_{opt}^{(b)}$ and $R_{opt}^{(b0)}$. For large E_b/N_0, they are the same. This fact can be explained as follows. It has been shown [21,22] that for small R, $E_b(R) = E_{b0}(R)$. As E_b/N_0 becomes large we have seen that both $R_{opt}^{(b)}$ and $R_{opt}^{(b0)}$ becomes small. Therefore, for large E_b/N_0, the optimal code rates found in terms of

both $E_b(R)$ and $E_{b0}(R)$ must be the same.

4.2.2. The Optimal Code Rate in Terms of the Minimum Distance Bound

For an (n,k) binary block code with minimum distance d_{min}, we define the asymptotic coding gain in the hard quantization case [see Appendix C] as

$$\gamma = 10 \log_{10} \frac{R \cdot d_{min}}{2} \text{ dB}, \tag{4.23}$$

for large E_b/N_0. The asymptotic lower and upper bounds on d_{min} are [43]

$$\lim_{n \to \infty} \frac{d_{min}}{n} > H^{-1}(1-R), \tag{4.24}$$

where $H(x) = -x \log_2 x - (1-x)\log_2(1-x)$, for $0 \leq x \leq \frac{1}{2}$, is the binary entropy function, and

$$\lim_{n \to \infty} \frac{d_{min}}{n} \leq \frac{1}{2}(1-R), \tag{4.25}$$

respectively.

Define

$$F_b(R) = R \cdot \lim_{n \to \infty} \frac{d_{min}}{n} . \tag{4.26}$$

From (4.23) we see that the optimal code rate in terms of the minimum distance bound, denoted by $R_{opt}^{(bd)}$, should maximize (4.26). By numerical calculation, we find that

$$R_{opt}^{(bd)} = \begin{cases} 0.3778, & \text{from the lower bound (4.24)}, \\ 0.5 \quad , & \text{from the upper bound (4.25)} \end{cases} \tag{4.27}$$

These results show that for large E_b/N_0, the optimal code rate

$R_{opt}^{(bd)} \in [0.3778, 0.5]$.

An apparently contradictory fact is that the optimal code rate found in terms of the error probability bounds is a function of E_b/N_0, while the optimal code rate found by using minimum distance bounds is a fixed number for large E_b/N_0. If one notice that the coding gain of (4.23) are defined by using only the first term of the decoding error probability expression [see Appendix C], the data become comprehensive.

4.3. The Optimal Convolutional Code Rate

4.3.1. The Optimal Code Rate in Terms of the Error Probability Bound

The channel coding theorem for convolutional codes states that [21]: For a Q-ary input DMC, there exists an (n,k,m) time-varying convolutional code of constraint length $n_A = n(1+m)$, and arbitrary sequence length, whose bit error probability P_b, resulting from maximum-likelihood decoding, is bounded by

$$P_b \le (2^k-1) \frac{2^{-n_A E_c(R)}}{[1-2^{-\delta n E_c(R)}]^2}, \quad \delta > 0, \tag{4.28.1}$$

$$E_c(R) = R_0 \quad , \quad 0 \le R \le R_0(1-\delta), \tag{4.28.2}$$

$$\begin{cases} E_c(R) = \max\limits_{\underline{q}} E_0(\rho, \underline{q}) \quad , \quad 0 \le \rho \le 1, \\ \\ R = (1-\delta)\max \dfrac{E_0(\rho,\underline{q})}{\rho} \; , \; R_0(1-\delta) \le R \le C(1-\delta), \end{cases} \tag{4.28.3}$$

where δ is an arbitrary positive number, and $E_0(\rho, \underline{q})$ and R_0 are given in (4.1) and (4.8), respectively. We define the optimal code rate $R_{opt}^{(c)}$

as the code rate R which

$$\text{maximizes:} \quad E_c(R),$$

$$\text{subject to:} \quad E_c(R) > 0. \tag{4.29}$$

If the value of R is restricted only to $0 \leq R \leq R_0(1-\delta)$, from (4.28.1) and (4.28.2) we obtain

$$P_b \leq (2^k-1) \frac{2^{-n_A E_{c0}(R)}}{[1-2^{-\delta n E_{c0}(R)}]^2} \quad , \quad \delta > 0, \tag{4.30.1}$$

$$E_{c0}(R) = R_0 \quad , \quad 0 \leq R \leq R_0(1-\delta). \tag{4.30.2}$$

In this case the optimization problem is

$$\text{maximize:} \quad E_{c0}(R),$$

$$\text{subject to:} \quad E_{c0}(R) > 0, \tag{4.31}$$

and the code rate R which satisfies (4.31) is denoted by $R_{opt}^{(c0)}$.

For most real channels, $E_0(\rho, \underline{q})$ (and consequently R_0) is an increasing function of $E_N/N_0 = R \cdot E_b/N_0$. Therefore, without loss of generality, we can make the following assumption:

> $E_0(\rho, \underline{q})$ (and R_0) is an increasing function of R for fixed E_b/N_0 and an increasing function of E_b/N_0 for fixed R.

Under this assumption, the optimization problems of (4.29) and (4.31) can be reduced to:

$$R_{opt}^{(c)} = \max_{0 \leq R \leq C} R,$$

$$\text{subject to:} \quad E_c(R) > 0, \tag{4.32}$$

and

$$R_{opt}^{(c0)} = \max_{0 \le R \le R_0} R,$$

$$\text{subject to: } E_{c0}(R) > 0, \tag{4.33}$$

respectively.

Note that $R_{opt}^{(c)}$ is always greater than or equal to $R_{opt}^{(c0)}$. Also, by our assumption, $E_c(R)$ and $E_{c0}(R)$ are increasing functions of E_b/N_0 for fixed R, and this implies that the optimal code rate is proportional to E_b/N_0. Because $E_c(R)$ and $E_{c0}(R)$ approach $\log_2 Q$ on a Q-ary input DMC as E_b/N_0 goes to infinity, the optimal code rate $R_{opt}^{(c)}$ (or $R_{opt}^{(c0)}$) also approaches $\log_2 Q$ asymptotically. The optimal code rates as a function of E_b/N_0 for the same modulation schemes considered for block codes are given in Tables 4.5-4.8 and shown graphically in Figures 4.4-4.6.

Note that in Figures 4.4-4.6, the optimal convolutional code rate is very close to $\log_2 Q$ in the practically interesting range of E_b/N_0. This implies that no coding is necessary. The reason for obtaining such a rather surprising result is that the convolutional coding exponents $E_c(R)$ and $E_{c0}(R)$ are not as accurate as the block coding exponents $E_b(R)$ and $E_{b0}(R)$ at low code rates. To obtain a more meaningful optimal code rate, a tighter bound on decoding error probability at low code rates - the expurgated upper bound - should be used. The expurgated bound for convolutional codes states that [21]: For binary-input, output-symmetric channels, there exists a time-varying convolutional code of constraint length n_A and rate $R = k/n$ bits per channel use for which the bit error probability with maximum likelihood decoding satisfies

Table 4.5. $R_{opt}^{(c)}$ for BPSK Modulation over an AWGN Channel with Output Hard Quantization

E_b/N_0	$R_{opt}^{(c)}$	C	$E_c(R_{opt}^{(c)})$
1.0 (T)	0	0	0
1.1	0.01	0.0101	0.0001
1.2	0.09	0.0945	0.0048
1.3	0.15	0.1641	0.0159
1.5	0.26	0.3015	0.0525
2.0	0.42	0.5391	0.1924
2.5	0.51	0.6921	0.3652
3.0	0.65	0.7888	0.5511
3.5	0.70	0.8973	0.7011
4.0	0.79	0.9473	0.7933
4.5	0.85	0.9719	0.8541
5.0	0.89	0.9845	0.8950
5.5	0.92	0.9913	0.9239
6.0	0.94	0.9950	0.9440
6.5	0.95	0.9970	0.9578
7.0	0.96	0.9982	0.9683
7.5	0.97	0.9990	0.9764
8.0	0.98	0.9994	0.9824
9.5	0.99	0.9999	0.9923

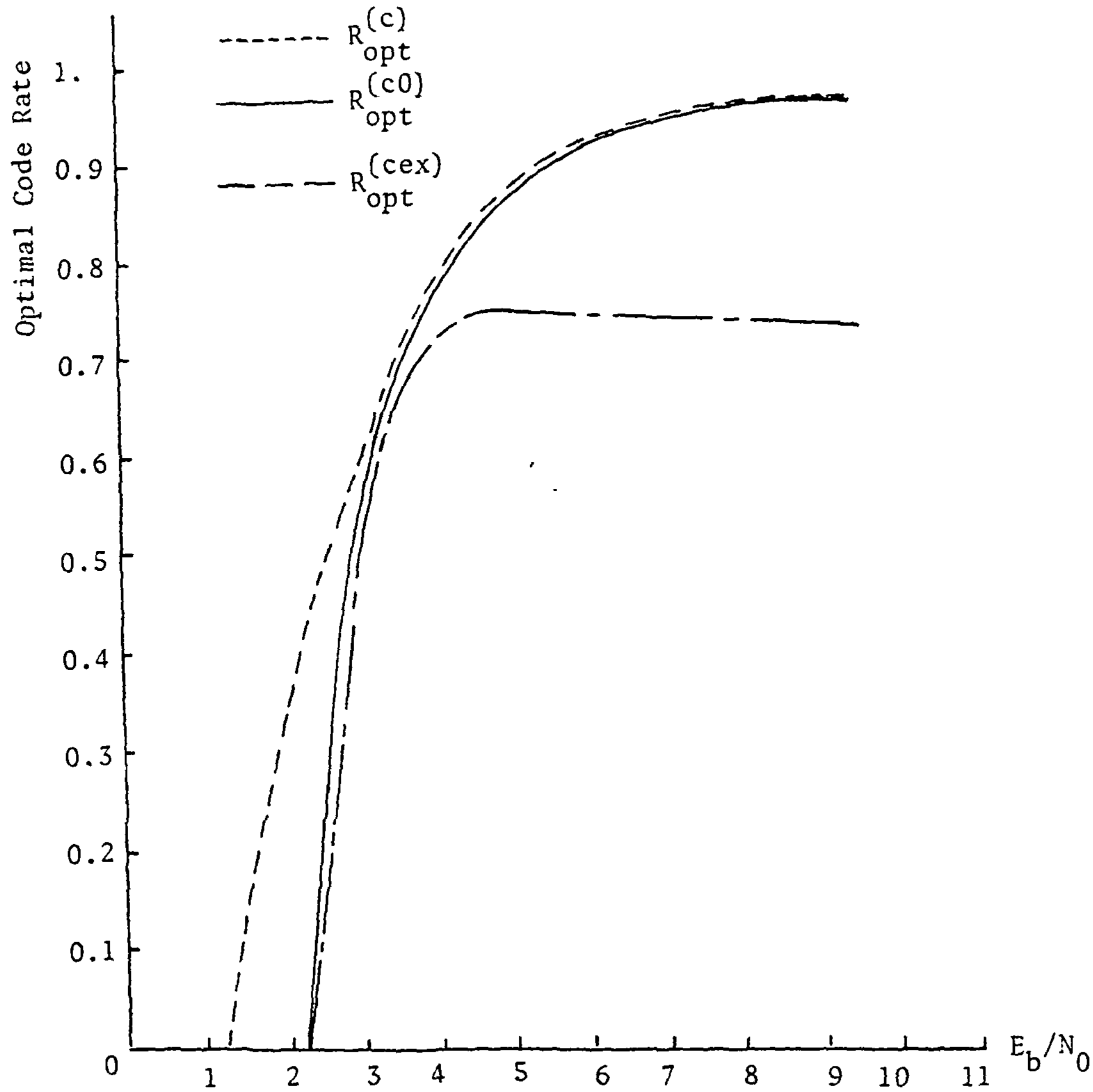

Figure 4.4. Optimal Time Varying Convolutional Code Rate for BPSK Modulation over an AWGN Channel with Output Hard Quantization

Table 4.6(a). $R_{opt}^{(c0)}$ for BPSK Modulation over an AWGN Channel with no Output Quantization

E_b/N_0	$R_{opt}^{(c0)}$	$E_{c0}(R_{opt}^{(c0)})(R_0)$
1.3 (T)	0	0
1.4	0.02	0.0201
1.5	0.20	0.2002
1.7	0.44	0.4409
2.0	0.65	0.6523
2.3	0.77	0.7733
2.5	0.82	0.8253
3.0	0.90	0.9062
3.5	0.94	0.9472
4.0	0.97	0.9705
4.5	0.98	0.9826
5.5	0.99	0.9938

Table 4.6(b). $R_{opt}^{(c0)}$ for QPSK Modulation over an AWGN Channel with no Output Quantization

E_b/N_0	$R_{opt}^{(c0)}$	$E_{c0}(R_{opt}^{(c0)})(R_0)$
1.3 (T)	0	0
1.4	0.04	0.0402
1.5	0.40	0.4004
1.7	0.88	0.8818
2.0	1.30	1.3046
2.3	1.54	1.5466
2.5	1.64	1.6506
3.0	1.80	1.8124
3.5	1.88	1.8944
4.0	1.94	1.9410
4.5	1.96	1.9652
5.5	1.98	1.9876

Table 4.6(c). $R_{opt}^{(c0)}$ for 8-ary PSK Modulation over an AWGN Channel with no Output Quantization

E_b/N_0	$R_{opt}^{(c0)}$	$E_{c0}(R_{opt}^{(c0)})(R_0)$
1.3 (T)	0	0
1.4	0.03	0.0303
1.5	0.39	0.3912
1.8	1.05	1.0575
2.0	1.35	1.3527
2.5	1.77	1.7838
3.0	2.04	2.0562
3.5	2.25	2.2569
4.0	2.40	2.4072
4.5	2.52	2.5275
5.0	2.61	2.6223
5.5	2.70	2.7027
6.0	2.76	2.7639
6.5	2.79	2.8095
7.0	2.85	2.8521
8.0	2.88	2.904
9.0	2.94	2.9412

Table 4.6(d). $R_{opt}^{(c0)}$ for 16-ary PSK Modulation over an AWGN Channel with no Output Quatnization

E_b/N_0	$R_{opt}^{(c0)}$	$E_{c0}(R_{opt}^{(c0)})(R0)$
1.3 (T)	0	0
1.4	0.04	0.0404
1.5	0.40	0.4004
1.7	0.88	0.8840
2.0	1.32	1.3332
2.5	1.76	1.7788
3.0	2.04	2.0564
3.5	2.24	2.2556
4.0	2.40	2.4136
4.5	2.52	2.5412
5.0	2.64	2.656
5.5	2.76	2.7612
6.0	2.84	2.8472
6.5	2.92	2.9276
7.0	3.00	3.0024
7.5	3.04	3.0632
8.0	3.12	3.1300
9.0	3.24	3.2436

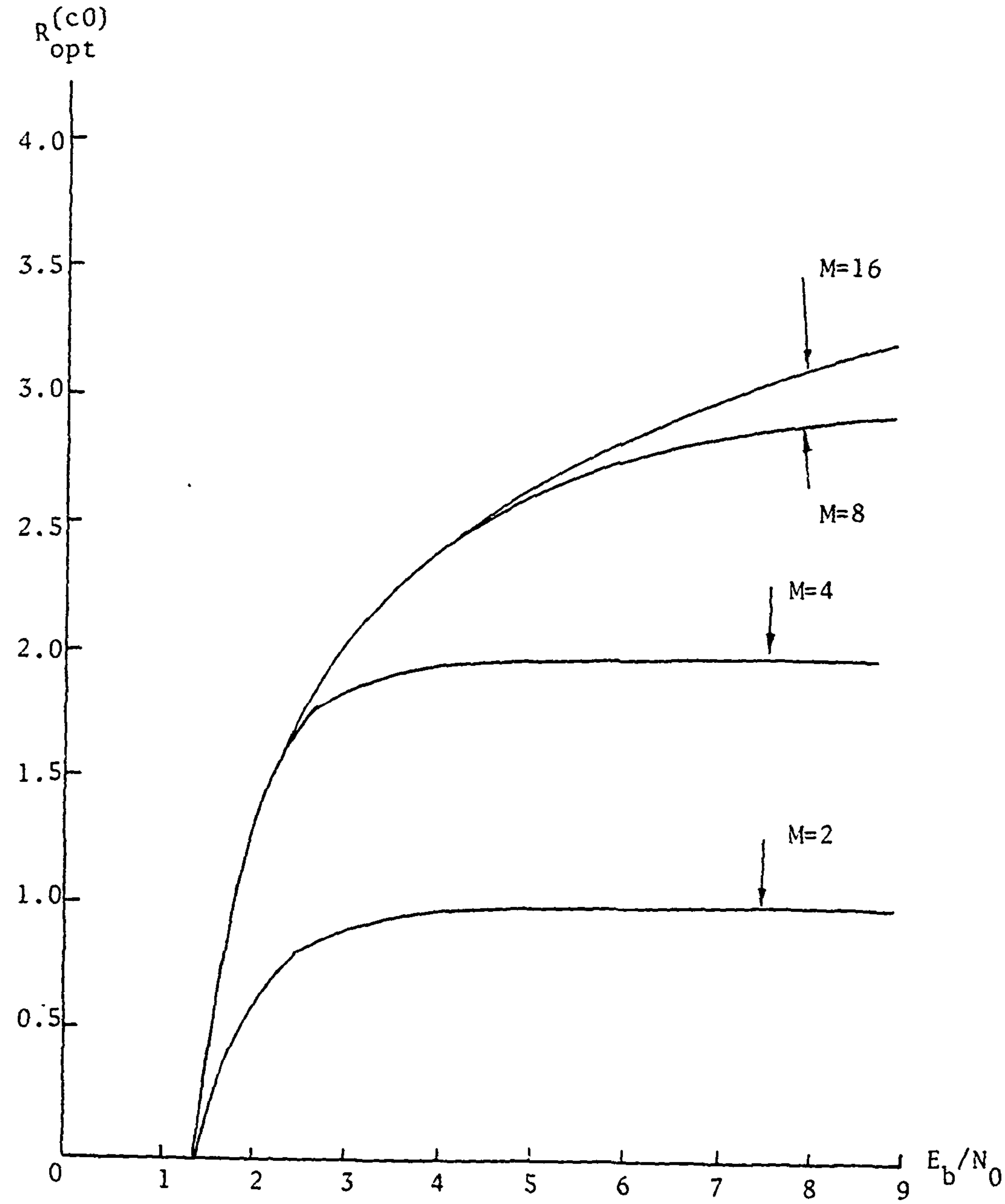

Figure 4.5. Optimal Time Varying Convolutional Code Rate for MPSK Modulation over an AWGN Channel with no Output Quantization

Table 4.7. $R_{opt}^{(c0)}$ for BPSK Modulation over an AWGN Channel with Output Hard Quantization

E_b/N_0	$R_{opt}^{(c0)}$	$E_{c0}(R_{opt}^{(c0)})(R_0)$
2.1 (T)	0	0
2.2	0.02	0.0201
2.5	0.28	0.2810
3.0	0.55	0.5503
3.5	0.70	0.7011
4.0	0.79	0.7933
4.5	0.85	0.8541
5.0	0.89	0.8950
6.0	0.94	0.9440
6.5	0.95	0.9578
7.0	0.96	0.9683
7.5	0.97	0.9764
8.0	0.98	0.9824
9.5	0.99	0.9923

Table 4.8(a). $R_{opt}^{(c0)}$ for BFSK Modulation over an AWGN Channel with Output Hard Quantization

E_b/N_0	$R_{opt}^{(c0)}$	$E_{c0}(R_{opt}^{(c0)})(R_0)$
9.5 (T)	0	0
10.0	0.64	0.6409
10.5	0.72	0.7221
11.0	0.77	0.7740
11.5	0.81	0.8142
12.0	0.84	0.8448
13.0	0.89	0.8912
14.0	0.92	0.9207
15.0	0.94	0.9411

Table 4.8(b). $R_{opt}^{(c0)}$ for QFSK Modulation over an AWGN Channel with Output Hard Quantization

E_b/N_0	$R_{opt}^{(c0)}$	$E_{c0}(R_{opt}^{(c0)})(R_0)$
5.5 (T)	0	0
6.0	1.40	1.4012
6.5	1.60	1.6086
7.0	1.72	1.7288
7.5	1.80	1.8064
8.0	1.86	1.8602
8.5	1.88	1.8926
9.0	1.92	1.9214
10.0	1.94	1.9532
11.0	1.96	1.9726
12.0	1.98	1.9840

Table 4.8(c). $R_{opt}^{(c0)}$ for 8-ary FSK Modulation over an AWGN Channel with Output Hard Quantization

E_b/N_0	$R_{opt}^{(c0)}$	$E_{c0}(R_{opt}^{(c0)})(R_0)$
4.5 (T)	0	0
5.0	2.46	2.4639
5.5	2.67	2.6817
6.0	2.79	2.8011
6.5	2.85	2.8695
7.0	2.91	2.9163
7.5	2.94	2.9442
8.5	2.97	2.9745

Table 4.8(d). $R_{opt}^{(c0)}$ for 16-ary FSK Modulation over an AWGN Channel with Output Hard Quantization

E_b/N_0	$R_{opt}^{(c0)}$	$E_{c0}(R_{opt}^{(c0)})(R_0)$
3.5 (T)	0	0
4.0	2.76	2.7648
4.5	3.52	3.5208
5.0	3.72	3.7404
5.5	3.82	3.8556
6.0	3.88	3.9140
6.5	3.92	3.9496
7.0	3.96	3.9708

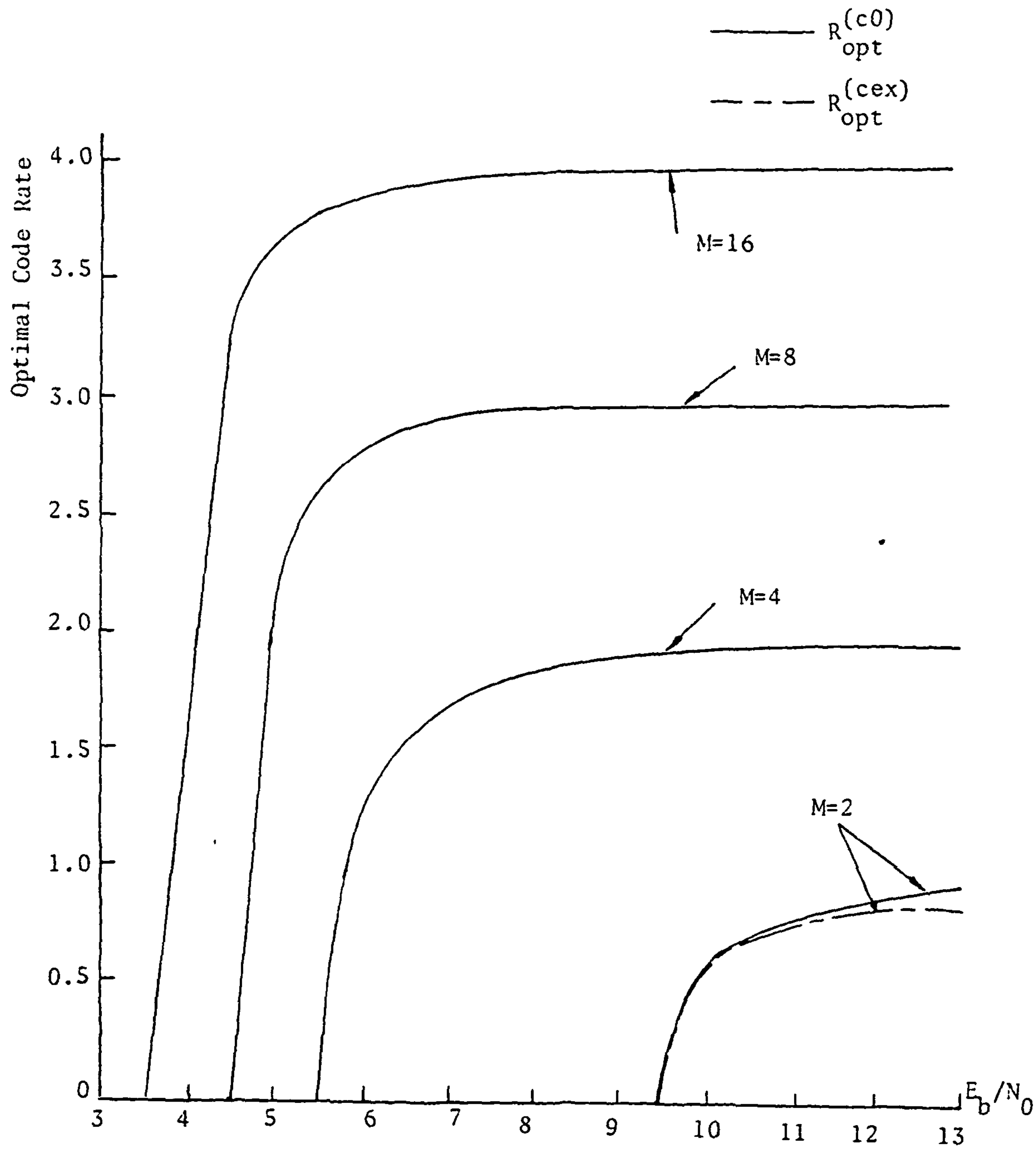

Figure 4.6. Optimal Time Varying Convolutional Code Rate for MFSK Modulation over an AWGN Channel with Output Hard Quantization

$$P_b < \left[\frac{2^k-1}{(1-2^{-b\delta})^2}\right]^{E_{cex}(R)/[R(1+\delta)]} 2^{-n_A E_{cex}(R)} , \tag{4.34.1}$$

where

$$E_{cex}(R) = \frac{R(1+\delta)\ln Z}{\ln[2^{-R(1+\delta)}-1]} , \quad 0 \le R \le R_0/(1+\delta), \tag{4.34.2}$$

$\delta > 0$ is an arbitrary positive number, and

$$Z = \sum_y \sqrt{P(y|0)p(y|1)} . \tag{4.34.3}$$

Since the exponential term dominates the error probability expression, the optimal code rate, denoted by $R_{opt}^{(cex)}$, satisfies

$$\begin{aligned} &\text{maximizing:} \quad E_{cex}(R), \\ &\text{subject to:} \quad E_{cex}(R) > 0. \end{aligned} \tag{4.35}$$

The optimal code rates as a function of E_b/N_0, for BPSK and BFSK modulation on an AWGN channel with output hard quantization are shown in Tables 4.9 and 4.10 and plotted in Figures 4.4 and 4.6, respectively. Note that $R_{opt}^{(cex)}$ is around 0.75 for large E_b/N_0.

4.3.2. The Optimal Code Rate in Terms of the Free Distance Bound

With maximum-likelihood decoding of convolutional codes, for large E_b/N_0, the asymptotic coding gain is given by [2]

$$\gamma = 10 \log_{10}\frac{R\cdot d_{free}}{2} \text{ dB} , \quad \text{for hard quantization,} \tag{4.36}$$

Table 4.9. $R_{opt}^{(cex)}$ for BPSK Modulation over an AWGN Channel with Output Hard Quantization

E_b/N_0	$R_{opt}^{(cex)}$	R_0	$E_{cex}(R_{opt}^{(cex)})$
2.2 (T)	0.02	0.0201	0.0201
2.3	0.12	0.1203	0.1203
2.4	0.21	0.2102	0.2102
2.5	0.28	0.2810	0.2811
2.6	0.35	0.3506	0.3507
2.7	0.41	0.4104	0.4105
2.8	0.46	0.4610	0.4612
2.9	0.50	0.5028	0.5038
3.0	0.55	0.5503	0.5505
3.2	0.62	0.6204	0.6206
3.4	0.67	0.6734	0.6758
3.6	0.72	0.7222	0.7241
4.0	0.75	0.7753	0.8021
5.0	0.75	0.8482	0.9906
10.0	0.75	0.9790	1.9019
15.0	0.75	0.9970	2.7881

Table 4.10. $R_{opt}^{(cex)}$ for BFSK Modulation over an AWGN Channel with Output Hard Quantization

E_b/N_0	$R_{opt}^{(cex)}$	R_0	$E_{cex}(R_{opt}^{(cex)})$
9.65 (T)	0.51	0.5102	0.5102
9.70	0.54	0.5408	0.5412
9.90	0.62	0.6201	0.6201
10.0	0.64	0.6410	0.6415
10.3	0.69	0.6925	0.6944
10.5	0.72	0.7221	0.7239
10.8	0.76	0.7601	0.7605
11.0	0.77	0.7740	0.7784
11.5	0.79	0.8040	0.8215
15.0	0.78	0.8945	1.1155
20.0	0.78	0.9593	1.5369
25.0	0.77	0.9835	1.9590
30.0	0.77	0.9937	2.3811
35.0	0.77	0.9976	2.8033
40.0	0.77	0.9991	3.2255

$$\gamma = 10 \log_{10} R \cdot d_{free} \text{ dB} \quad , \text{ for infinite fine quantization} \tag{4.37}$$

where d_{free} is the free distance of the convolutional code. The asymptotic bounds on free distance are given by [44]:

$$\lim_{n_A \to \infty} \frac{d_{free}}{n_A} < \frac{1-R}{2} , \tag{4.38.1}$$

$$\lim_{n_A \to \infty} \frac{d_{free}}{n_A} \geq H^{-1}(1-R), \tag{4.38.2}$$

for systematic fixed codes, where n_A is the code constraint length;

$$\lim_{n_A \to \infty} \frac{d_{free}}{n_A} < \frac{1}{2} , \tag{4.39.1}$$

$$\lim_{n_A \to \infty} \frac{d_{free}}{n_A} \geq \begin{cases} \dfrac{2R(1 - 2^{2R-1})}{H(2^{2R-1}) + 2R-1} , & 0 \leq R \leq \dfrac{3}{8} \\[2ex] 2H^{-1}(1-R) , & \dfrac{3}{8} \leq R \leq 1, \end{cases} \tag{4.39.2}$$

for nonsystematic fixed codes; and

$$\lim_{n_A \to \infty} \frac{d_{free}}{n_A} \geq \frac{R(1 - 2^{R-1})}{H(2^{R-1}) + R-1} , \tag{4.40}$$

for nonsystematic time-varying codes.

Define

$$F_c(R) = R \cdot \lim_{n_A \to \infty} \frac{d_{free}}{n_A} . \tag{4.41}$$

The code rate R which maximizes $F_c(R)$ is called the optimal code rate, and is denoted by $R_{opt}^{(cd)}$. If follows from (4.36) and (4.37) that $R_{opt}^{(cd)}$

is the code rate which maximizes the coding gain for large E_b/N_0.

The optimization probelm can best be solved by computer. By numerical calculation, we find that

1) For systematic fixed codes

$$R_{opt}^{(cd)} = \begin{cases} 0.5 & \text{, from the upper bound (4.38.1),} \\ 0.3778, & \text{from the lower bound (4.38.2);} \end{cases} \tag{4.42}$$

2) For nonsystematic fixed codes

$$R_{opt}^{(cd)} = \begin{cases} 1 & \text{, from the upper bound (4.39.1),} \\ 0.3778, & \text{from the lower bound (4.39.2);} \end{cases} \tag{4.43}$$

3) For nonsystematic time varying codes

$$R_{opt}^{(cd)} = 0.7580, \text{ from the lower bound (4.40).} \tag{4.44}$$

These results indicate that for systematic fixed codes low code rate should be used while for nonsystematic codes high code rate should be used for optimum performance. It is interesting to compare (4.44) with the optimal code rate $R_{opt}^{(cex)}$ found in terms of the expurgated bound, they both indicate that the optimal code rate is around 0.75 for large E_b/N_0.

4.4. The Optimal Concatenated Code Rate

The complexity of conventional coding systems grows exponentially with the block length for block codes or with the constraint length for convolutional codes. To overcome the complexity of very long codes,

Forney [24] first introduced concatenated codes as a practical means of implementing codes with long block or constraint lengths. Concatenated coding systems are usually implemented by employing two levels of coding, as illustrated in Figure 4.7. Binary data from the data source is serially partitioned into k_1-bit blocks that are subsequently used as input signals to a 2^{k_1}-ary block encoder known as the outer encoder. Usually $Q = 2^{k_1}$-ary RS codes are used for this purpose. The output of the RS encoder is converted back into bits and serially encoded by the inner encoder, which may be either block or convolutional (only block inner codes are considered here), and the resultant sequence of channel symbols is sent over the physical channel - the inner channel. Decoding is accomplished in the reverse order.

To evaluate the optimal code rate of the concatenated code described above, let us investigate the performance obtainable with the following concatenation scheme. The inner code is a block code of length n_1 and rate R_1, where $R_1 = \log_2 M/n_1$, and M is the number of code words in the inner code. The inner decoder is a maximum-likelihood decoder which puts out an estimate of which code word was sent, with average probability of error p. The outer code is a block code of length n_2 and rate R_2. Given an estimate, the outer decoder makes no distinction between the outputs other than the estimate. Therefore, we regard the outer channel as an equierror channel, and the outer channel has transition probability matrix

$$p(y|x) = \begin{cases} 1-p & y = x, \\ p(2^{n_1 R_1}-1)^{-1}, & y \neq x. \end{cases} \tag{4.45}$$

The overall rate and length of the concatenated code are $R = R_1 R_2$

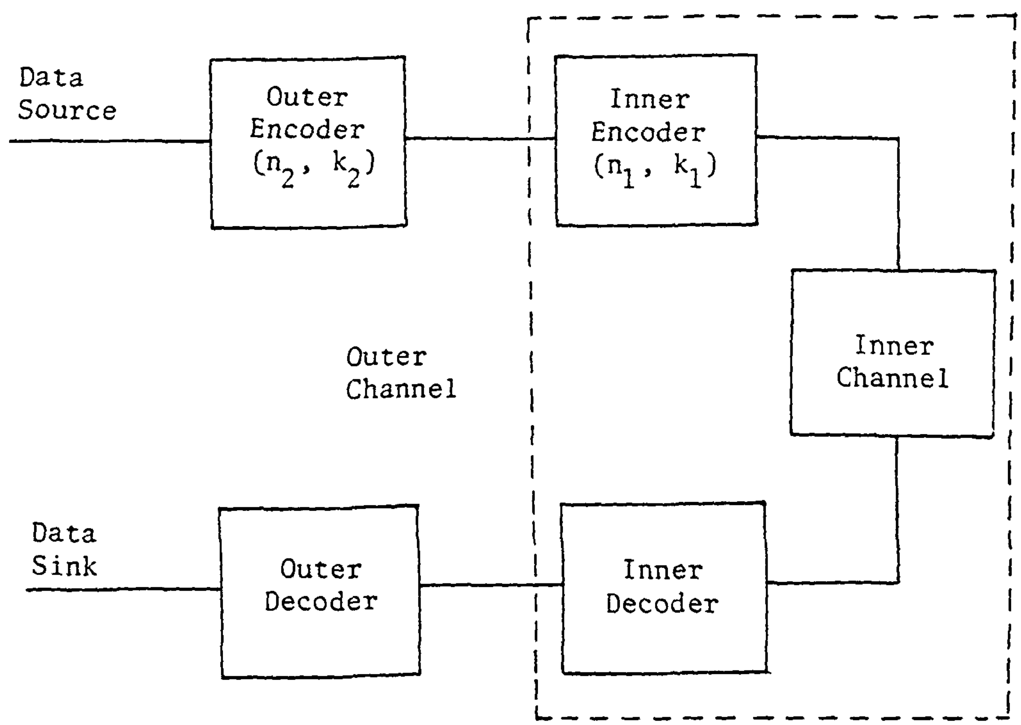

Figure 4.7. Concatenated Coding System

and $N = n_1 n_2$, respectively. Define $R^* = n_1 R$. Note that the total number of code words in the concatenated code is $2^{n_2 R_2 n_1 R_1} = 2^{n_2 R^*} = 2^{NR}$.

First, we derive a random coding bound for the concatenated code in terms of the channel cutoff rate. Suppose that the best inner code of length n_1 and rate R_1 is used. Then

$$p \leq 2^{-n_1(R_{01}-R_1)} \quad , \quad 0 \leq R_1 \leq R_{01}, \tag{4.46}$$

where R_{01} is the inner channel cutoff rate. The outer channel cutoff rate, denoted by R_{02}, is given by (4.8). By symmetry, the probability vector $\underline{q}$ for which the left hand side of (4.8) is maximized is

$$q(x) = 2^{-n_1 R_1} \quad , \quad \text{for all } x. \tag{4.47}$$

Substituting (4.45-4.47) into (4.8), we obtain

$$R_{02} \geq -\log_2 2^{-n_1 R_1} \cdot \left[\sqrt{1 - 2^{-n_1(R_{01}-R_1)}} + \sqrt{(2^{n_1 R_1}-1)\, 2^{-n_1(R_{01}-R_1)}}\right]^2 . \tag{4.48}$$

As n_1 becomes large, (4.48) is reduced to

$$R_{02} \geq \begin{cases} n_1 R_1 & R_{01} \geq 2R_1, \\ n_1(R_{01}-R_1) \quad , & R_{01} \leq 2R_1, \end{cases}$$

or in another form

$$R_{02} \geq n_1 \min[(R_{01}-R_1), R_1]. \tag{4.49}$$

Now define

C-2

$$
\begin{aligned}
E_{n0}^*(R^*) &= R_{02} - R^* \\
&\geq n_1 \min[(R_{01}-R_1), R_1] - n_1 R \\
&= n_1\{\min[(R_{01}-R_1), R_1] - R\}.
\end{aligned} \tag{4.50}
$$

In order to maximize performance, for a given overall rate R, R_1 and R_2 may be varied subject to the constraint $R = R_1 R_2$. We then define

$$
\begin{aligned}
E_{n0}(R) &= \frac{1}{n_1} \max_{R_1 R_2 = R} E_{n0}^*(R^*) \\
&\geq \max_{R_1 R_2 = R} \{\min[(R_{01}-R_1), R_1] - R\}.
\end{aligned} \tag{4.51}
$$

Therefore, we can claim from the coding theorem (in terms of the cutoff rate) that if there exists an equierror outer channel with $2^{n_1 R_1}$ inputs and average error probability p, then there exists a concatenated code of overall length N and overall rate R such that

$$
P_E \leq 2^{-NE_{n0}(R)}, \quad 0 \leq R \leq R_{01}. \tag{4.52}
$$

Note that in (4.51), R_{01} is a function of R_1 and R_2, and so is $E_{n0}(R)$. To emphasize this fact, we rewrite $E_{n0}(R)$ as $E_{n0}(R_1, R_2)$, and consequently

$$
P_E \leq 2^{-NE_{n0}(R_1, R_2)}, \quad 0 \leq R \leq R_{01}. \tag{4.53}
$$

Now the optimal inner and outer code rates, denoted by $R_{opt}^{(10)}$ and $R_{opt}^{(20)}$, respectively, should satisfy

$$
\begin{aligned}
\text{maximize: } & E_{n0}(R_1, R_2), \\
\text{subject to: } & E_{n0}(R_1, R_2) > 0.
\end{aligned} \tag{4.54}
$$

A tighter bound on P_E is given by a concatenated coding theorem in Reference 24. But in [24] the inner code rate is restricted to $R_{c1} \leq R_1 \leq C_1$, where R_{c1} and C_1 are the inner channel critical rate and the inner channel capacity, respectively. For our optimal code rate problem, the inner code rate can be any value between 0 and C_1. This is done by slightly modifying Forney's theorem in [24]. From the block coding theorem, the outer channel average probability of error is bounded by

$$p \leq 2^{-n_1 E_b(R_1)} \quad , \quad 0 \leq R_1 < C_1, \tag{4.55}$$

where $E_b(R_1)$ is given by (4.5). Using (4.55) and following the same line of reasoning as Forney, we obtain: If there exists an equierror outer channel with $2^{n_1 R_1}$ inputs and average error probability p, bounded by (4.55), then there exists a concatenated code of overall length N and overall rate R such that

$$P_E \leq 2^{-NE_n(R)} \quad , \quad 0 \leq R \leq C_1, \tag{4.56}$$

where

$$E_n(R) = \max_{R_1 R_2 = R} \{\min[E_b(R_1), R_1](1-R_2)\}. \tag{4.57}$$

Again note that $E_n(R)$ is, in fact, a function of R_1 and R_2, so we can rewrite it as $E_n(R_1, R_2)$, and (4.56) becomes

$$P_E \leq 2^{-NE_n(R_1, R_2)} \quad , \quad 0 \leq R \leq C_1. \tag{4.58}$$

Therefore the optimization problem is

$$\text{maximize:}\quad E_n(R_1, R_2),$$

$$\text{subject to:}\quad E_n(R_1, R_2) > 0, \tag{4.59}$$

and the inner and outer code rates R_1 and R_2 which satisfy (4.59) are denoted by $R_{opt}^{(1)}$ and $R_{opt}^{(2)}$, respectively.

Example 4.3

Assume that the inner channel is a BSC derived from forcing hard decisions on an AWGN channel with BPSK modulation. This channel is representative of the deep space channel where concatenated codes have met with a great deal of success. The numerical values of optimal inner and outer code rates in terms of both $E_{n0}(R_1, R_2)$ and $E_n(R_1, R_2)$ are listed in Table 4.11 and shown in Figure 4.8 as a function of E_b/N_0. From Figure 4.8 we see that low inner code rates and high outer code rates should be used in such a concatenated coding system.

Table 4.11(a). $R_{opt}^{(10)}$ and $R_{opt}^{(20)}$ for BPSK Modulation over an AWGN Channel with Output Hard Quantization

E_b/N_0	$R_{opt}^{(10)}$	$R_{opt}^{(20)}$	R_{01}	$E_{n0}(R_{opt}^{(10)}, R_{opt}^{(20)})$
4.4	0.01	1.0	0.0200	2×10^{-6}
4.5	0.02	0.98	0.0398	0.0002
4.7	0.04	0.96	0.0809	0.0016
4.9	0.06	0.94	0.1204	0.0036
5.1	0.08	0.92	0.1604	0.0064
5.3	0.10	0.90	0.1999	0.0099
5.5	0.12	0.89	0.2410	0.0132
5.7	0.14	0.87	0.2790	0.0172
6.0	0.16	0.85	0.3202	0.0240
6.4	0.18	0.82	0.3615	0.0324
6.7	0.20	0.80	0.4002	0.0400
7.1	0.22	0.78	0.4420	0.0484
7.6	0.25	0.76	0.5013	0.0600
7.9	0.26	0.74	0.5200	0.0675
8.5	0.27	0.70	0.5404	0.0810
9.1	0.30	0.69	0.6005	0.0930
9.7	0.31	0.66	0.6205	0.1054
10.2	0.32	0.64	0.6402	0.1152
11.1	0.33	0.60	0.6597	0.1317
11.8	0.34	0.58	0.6815	0.1428
12.8	0.36	0.56	0.7206	0.1584

Table 4.11(b). $R_{opt}^{(1)}$ and $R_{opt}^{(2)}$ for BPSK Modulation over an AWGN Channel with Output Hard Quantization

E_b/N_0	$R_{opt}^{(1)}$	$R_{opt}^{(2)}$	C_1	$E_n(R_{opt}^{(1)}, R_{opt}^{(2)})$
1.2	0.05	0.98	0.0522	1×10^{-6}
1.3	0.08	0.96	0.0870	6×10^{-6}
1.4	0.11	0.94	0.1237	2.6×10^{-5}
1.5	0.14	0.93	0.1634	7.1×10^{-5}
1.6	0.16	0.91	0.1917	1.5×10^{-4}
1.7	0.18	0.90	0.2226	2.7×10^{-4}
1.8	0.20	0.89	0.2540	0.0004
2.0	0.22	0.86	0.2928	0.0009
2.1	0.24	0.86	0.3277	0.0012
2.3	0.26	0.84	0.3690	0.0021
2.5	0.28	0.82	0.4093	0.0031
2.8	0.30	0.80	0.4604	0.0051
3.2	0.32	0.78	0.5201	0.0084
3.8	0.34	0.75	0.5900	0.0147
4.3	0.35	0.73	0.6365	0.0211
5.0	0.36	0.71	0.6923	0.0312
5.75	0.37	0.69	0.7427	0.0435
6.75	0.38	0.67	0.7963	0.0613
7.75	0.38	0.64	0.8256	0.0795
8.75	0.38	0.61	0.8475	0.0971
9.75	0.38	0.58	0.8637	0.1138

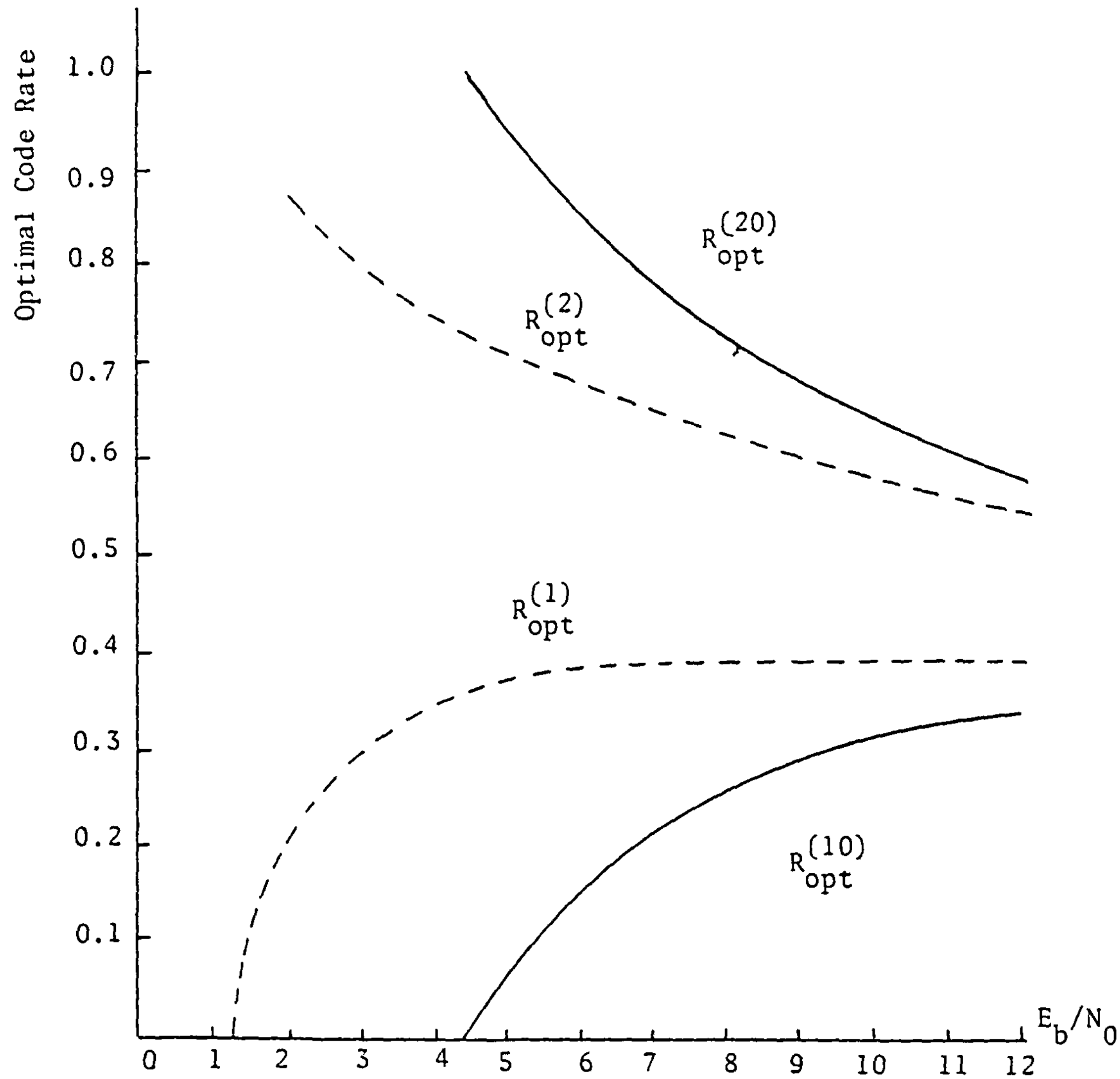

Figure 4.8. Optimal Concatenated Code Rate for BPSK Modulation over an AWGN Channel with Output Hard Quantization